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Analysis of Gutenberg-Richter *b*-value and m_{max} Part I: Exact Solution of Kijko-Sellevoll Estimator of m_{max}

Mika Haarala[1,2] y **Lía Orosco**[1,3]

Abstract

This report is the first of a series of three which have the main goal to achieve a method to estimate the parameters β and m_{max} that are essential when Gutenberg-Richter law is used for seismic hazard assessment.
We give the exact solution of Kijko-Sellevoll approach to estimateand also a new method to calculate m_{max} applying series. It proved to be numerically more stable even in the cases when wide range of magnitudes or large catalogue is used.

Keywords: Mmax - *b*-value - Gutenberg-Richter distribution function - Kijko-Sellevoll estimator - series

Resumen

Este es el primero de una serie de tres informes sobre un trabajo que tiene como principal objetivo lograr un método para estimar los parámetros β y m_{max}, que son esenciales cuando se utiliza la ley de Gutenberg – Richter para la estimación de la peligrosidad sísmica.
Proponemos la solución exacta del método de Kijko-Sellevoll para estimar m_{max} como así también mostramos un nuevo método para calcular m_{max} aplicando series. Este método es numéricamente más estable, aun en los casos en que se utiliza un catálogo sísmico en el que los valores se ubican en un intervalo amplio de magnitudes.

Palabras clave: Mmax - *b* - función de distribución Gutenberg-Richter - estimador de Kijko-Sellevoll - series

Introduction

In seismic hazard assessment studies, the very well-known frequency-magnitude distribution (Ishimoto, Iida, 1939; Gutenberg, Richter, 1944), commonly known as Gutenberg-Richter law,

$$\log_{10} N(M) = a - bM$$

[1] Instituto de Estudios Interdisciplinarios de Ingeniería (IESIING) – Facultad de Ingeniería – UCASAL
[2] Acrenet Oy, Joensuu, Finland.
[3] Facultad de Ingeniería. Universidad Nacional de Salta

is the cumulative number of events with magnitude grater or equal than M, a and b are some unknown constants to be determined by some method. When a probabilistic approach is used, the Gutenberg – Richter probabilistic density function

$$f(m) = \beta \exp\left[-\beta\left(m - m_{\min}\right)\right] \tag{1}$$

or the double truncated Gutenberg-Richter distribution function

$$f(m) = \frac{\beta \exp\left[-\beta\left(m - m_{\min}\right)\right]}{1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]} \tag{2}$$

are still applied and investigated (Anagnostopoulos et al., 2008; Ishibe, Shimazaki, 2008; Kahraman et al., 2008; Leyton et al., 2009; Amorèse et al., 2010; Holschneider et al., 2011; Zúñiga, Figueroa-Soto, 2012; Kijko, Smit, 2012; Mostafanejad et al., 2013; Rong et al., 2014; Márquez-Ramírez et al., 2015).

In (1) and (2) the β-value is related with b as $\beta = b\log(10)$, $m_{\min}$ is known like threshold of completeness of seismic catalogue and $m_{\max}$ is the maximum earthquake probably to occur.

In 1965, using moments of Gamma distribution function, Utsu (1965) derived a simple estimator for β-value, in the case of unbounded expression (1). Aki (1965) showed that this estimator is also a maximum likelihood estimator for the Gutenberg-Richter distribution function. In this work we will call it as Aki-Utsu estimator, which has been quite popular because of its simplicity.

Page (1968) proposed a maximum likelihood estimator for the bounded expression (2) which needs to be solved iteratively and it needs to estimate someway the parameter $m_{\max}$. Normally this is set to maximum observed magnitude.

Hamilton (1967), studying the stability of mean value and variance of sequences of earthquakes, used the moment method moments; later, Cosentino (Cosentino, Luzio, 1976; Cosentino et al., 1977) published the moment method estimators for the Gutenberg-Richter distribution function.

In1984, Kijko (1984) presented for the first time his idea to calculate the estimator of $m_{\max}$. Later, Kijko and Sellevoll (1989) developed the method itself using the double truncated Gutenberg-Richter distribution function (2). Kijko and Graham (1998), with basis in Cramer's approximation (Cramer, 1961), generalized this method making possible to apply it for different distribution functions. Kijko (2004) called the estimator of $m_{\max}$ of the Gutenberg-Richter model (2) as a «Kijko-Sellevoll (KS) estimator».

When applying Gutenberg-Richter model, parameter β must be defined a priori with some hypothesis; $m_{\max}$ is set as the maximum observed magnitude, as infinity or that determined by empirical formula when possible.

The objectives of this work are to show the algebraic solution to KS estimator, and to propose a method to estimate $m_{\max}$. This work is the first part of a series of three, which have the main goal to achieve a method to estimate β and $m_{\max}$ by using exact solution of Gutenberg-Richter law.

Exact solution of Kijko-Sellevoll estimator

Firstly we give the exact solution of KS model using exact value of β. Actually K-S estimator has two different solutions: first one (Kijko-Sellevoll function 1, named as KS-1) is a solution of the original KS model and the second one (Kijko-Sellevoll function 2, named as KS-2) is its counterpart such as $f_n^{KS-1} + f_n^{KS-2} = \beta(m_{\max} - m_{\min})$

In this work we are analyzing the double truncated Gutenberg-Richter distribution function (2) which has cumulative distribution function (CDF)

$$F_M(m \mid m_{\max}) = \begin{cases} 0, & \text{for } m < m_{\min}, \\ \dfrac{1-\exp\left[-\beta(m-m_{\min})\right]}{1-\exp\left[-\beta(m_{\max}-m_{\min})\right]} & \text{for } m_{\min} \le m < m_{\max}, \\ 1, & \text{for } m_{\max} \le m. \end{cases} \tag{3}$$

Because the unbounded limit of this distribution function exists

$$F_M(m \mid m_{\max} = \infty) = \lim_{m_{\max}\to\infty} F_M(m \mid m_{\max}) = \begin{cases} 0, & \text{for } m < m_{\min}, \\ 1-\exp\left[-\beta(m-m_{\min})\right] & \text{for } m_{\min} \le m, \end{cases}$$

and the lower limit exists

$$F_M(m \mid m_{\max} = m_{\min}) = \lim_{m_{\max}\to m_{\min}} F_M(m \mid m_{\max}) = \begin{cases} 0, & \text{for } m < m_{\min}, \\ 1, & \text{for } m_{\min} \le m, \end{cases}$$

we assume that $m_{\max} \in [m_{\min}, \infty]$

Let $M_1, M_2, \ldots, M_n \in [m_{\min}, m_{\max}]$ be a set of random variables (which we shall call catalogue C_n of size n). Let $M_{(1)} \le M_{(2)} \le \cdots \le M_{(n)}$ denote the ordered values of $M_1, M_2, \ldots, M_n$. That is to say, the random variable $M_{(n)}$ is a maximum in the catalogue C_n. We assume also that these random variables are independently and identically distributed (IID) with CDF $F_M(m)$ given by (3). Let $m_{(1)} \le m_{(2)} \le \cdots \le m_{(n)}$ be an ordered sample of magnitudes where $m_{(1)}$ is a minimum observed magnitude ($m_{\min} \le m_{(1)}$) and $m_{(n)}$ is a maximum observed magnitude ($m_{(n)} \le m_{\max}$). This $m_{(n)}$ has a CDF

$$F_{M_{(n)}}(m \mid m_{\max}) = \begin{cases} 0, & \text{for } m < m_{\min}, \\ \left[F_M(m \mid m_{\max})\right]^n & \text{for } m_{\min} \le m < m_{\max}, \\ 1, & \text{for } m_{\max} \le m. \end{cases} \tag{4}$$

Integrating by parts, the expected value of $M_{(n)}$ is

$$E\left(M_{(n)} \mid m_{\max}\right) = \int_{m_{\min}}^{m_{\max}} m \, dF_{M_{(n)}}\left(m \mid m_{\max}\right) = m_{\max} - \int_{m_{\min}}^{m_{\max}} F_{M_{(n)}}\left(m \mid m_{\max}\right) dm. \tag{5}$$

Then Kijko set

$$m_{\max} = E\left(M_{(n)} \mid m_{\max}\right) + \int_{m_{\min}}^{m_{\max}} F_{M_{(n)}}\left(m \mid m_{\max}\right) dm.$$

(In Appendix A we show the Kijko's method to find the estimator $\hat{m}_{\max}$.)

We define a new function

$$g\left(\mathfrak{M}\right) = \left[\mathfrak{M} - E\left(M_{(n)} \mid m_{\max}\right)\right] - \int_{m_{\min}}^{\mathfrak{M}} F_{M_{(n)}}\left(m \mid \mathfrak{M}\right) dm, \tag{6}$$

where $\mathfrak{M} \in \left[m_{\min}, \infty\right]$. For the estimator $\hat{m}_{\max}$ holds $g\left(\hat{m}_{\max}\right) = 0$ because it is a solution of equation (5). Function g is negative at the point $\mathfrak{M} = E\left(M_{(n)} \mid m_{\max}\right)$ since in equation (6) the first term is zero and, because of $F_{M_{(n)}}\left(m \mid \mathfrak{M}\right)$ is a positive function for any fixed n (i.e. $n < \infty$), the integral is positive.

The derivative of function g in (6) is (using Leibniz's theorem for differentiation of an integral; Abramowitz, Stegun, 1972)

$$\begin{aligned} g'\left(\mathfrak{M}\right) &= 1 - F_{M_{(n)}}\left(\mathfrak{M} \mid \mathfrak{M}\right) - \int_{m_{\min}}^{\mathfrak{M}} \frac{\partial}{\partial \mathfrak{M}} F_{M_{(n)}}\left(m \mid \mathfrak{M}\right) dm \\ &= -n \int_{m_{\min}}^{\mathfrak{M}} \left[F_M\left(m \mid \mathfrak{M}\right)\right]^{n-1} \frac{\partial}{\partial \mathfrak{M}} F_M\left(m \mid \mathfrak{M}\right) dm \geq 0 \end{aligned} \tag{7}$$

for all $\mathfrak{M} \in \left]m_{\min}, \infty\right[$ since $\partial_{\mathfrak{M}} F_M\left(m \mid \mathfrak{M}\right) \leq 0$. Thus g is a monotonically increasing function and because we showed above that there exists at least one point where the function g is negative, so it has at most one solution for $g\left(\hat{m}_{\max}\right) = 0$. This makes attractive the idea to apply the Newton-Raphson method to find the estimator $\hat{m}_{\max}$.

To find the solution of the last integral in equation (5)we write

$$\Delta = \int_{m_{\min}}^{m_{\max}} F_{M_{(n)}}\left(m \mid m_{\max}\right) dm = \frac{\int_{m_{\min}}^{m_{\max}} \left(1 - \exp\left[-\beta\left(m - m_{\min}\right)\right]\right)^n dm}{\left(1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]\right)^n}. \tag{8}$$

We can calculate the derivative for $\left(1 - \exp\left[-\beta\left(m - m_{\min}\right)\right]\right)^n$ as

$$-\frac{1}{\beta n}\frac{\partial}{\partial m}\left(1-\exp\left[-\beta\left(m-m_{\min}\right)\right]\right)^n=\left(1-\exp\left[-\beta\left(m-m_{\min}\right)\right]\right)^n$$
$$-\left(1-\exp\left[-\beta\left(m-m_{\min}\right)\right]\right)^{n-1},$$

which gives the integration formula

$$\int_{m_{\min}}^{m_{\max}}\left(1-\exp\left[-\beta\left(m-m_{\min}\right)\right]\right)^n dm=-\frac{1}{\beta n}\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^n$$
$$+\int_{m_{\min}}^{m_{\max}}\left(1-\exp\left[-\beta\left(m-m_{\min}\right)\right]\right)^{n-1} dm.$$

Applying this n times we can eliminate the power (on the last step the power is equal to zero) and the integral (8) can be expressed by

$$\Delta=\frac{1}{\beta}\frac{\beta\left(m_{\max}-m_{\min}\right)-\sum_{k=1}^{n}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^k}{k}}{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^n}. \tag{9}$$

To the logarithm function holds (Abramowitz, Stegun, 1972)

$$-\log\left(1-z\right)=\sum_{k=1}^{\infty}\frac{z^k}{k}, \qquad \left|z\right|\le 1,\ z\ne 1. \tag{10}$$

We can write now

$$\beta\left(m_{\max}-m_{\min}\right)=-\log\left[1-\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)\right].$$

Setting $z=1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]$ we see that conditions of (10) i.e. $0\le z<1$ holds when $m_{\max}\in\left[m_{\min},\infty\right[$ and applying this to equation (10) we have

$$\beta\left(m_{\max}-m_{\min}\right)=\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^k}{k}. \tag{11}$$

Hence the sum in (9) represents the first n terms of the series (11) so we can rewrite (9) in the form

$$\Delta=\frac{1}{\beta}\frac{\sum_{k=n+1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^k}{k}}{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^n}$$

Dividing each term by denominator and re-indexing the series we get the final result

$$\Delta = \frac{1}{\beta}\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{k}}{k+n}. \quad (12)$$

This is equal to the solution (9) when n is an integer. In the series (12) the variable n is continuous ($n \in \mathbb{R}$) and it has a derivative, whereas the solution (9) has not it. In Appendix B we give the method to calculate directly the numerical values for the series (12). In normal cases when n is small and $m_{\max} - m_{\min}$ is enough big, we can get numerical results of the series by using the solution (9). Any way it is recommended to use the method of series given in appendix B to avoid the numerical instability of the formula (9).

Now it is possible to rewrite the expression of expected value (5) as

$$\beta\left[m_{\max} - E\left(M_{(n)} \mid m_{\max}\right)\right] = \sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{k}}{k+n}. \quad (13)$$

The right side is the Kijko-Sellevoll function 1 (KS-1 or f_n^{KS-1}). In fact the actual form of the function is $y = f_n^{KS-1}(x)$ where $x = \beta\left(m_{\max} - m_{\min}\right)$ and $y = \beta\left[m_{\max} - E\left(M_{(n)} \mid m_{\max}\right)\right]$. Explanation of this is abstract. If we change the variable in the distribution function (3) setting $m = x/\beta + m_{\min}$, with $\beta > 0$ we get a normalized CDF

$$\breve{F}_X\left(x \mid x_{\max}\right) = \begin{cases} 0, & \text{for } x < 0, \\ \dfrac{1-\exp(-x)}{1-\exp\left(-x_{\max}\right)} & \text{for } 0 \le x < x_{\max}, \\ 1, & \text{for } x_{\max} \le x. \end{cases}$$

where $x_{\max} = \beta\left(m_{\max} - m_{\min}\right)$ is a pseudo maximum magnitude. This is the CDF of truncated exponential distribution function and the KS-1 function works in this space. It is to say that the KS function measures the relation between the pseudo maximum and the pseudo expected value of maximum.

We pointed out before that there is another KS function. Our goal was to establish if the derivative of g in the equation (6) can be written by using same series than in the case of KS-1. Fortunately we found out that it is possible to write g and g' using same series, for example by means of the use of Kummer's transformation (see Abramowitz and Stegun, 1972).

We start with the counterpart of the expected value given by (5), which we will write as

$$E\left(M_{(n)} \mid m_{\max}\right) = m_{\min} + \int_{m_{\min}}^{m_{\max}} 1 - F_{M_{(n)}}\left(m \mid m_{\max}\right) dm. \quad (14)$$

If the relation (5) measures the probability of occurrence, equation (14) measures the probability of non-occurrence. Because of $1-F_{M_{(n)}}(m \mid m_{\max})$ is positive function when $m_{\min} < m_{\max}$, then $E(M_{(n)} \mid m_{\max}) > m_{\min}$ for all fixed n and it is zero only if $m_{\min} = m_{\max}$.

Solving the integral we get (14)

$$E\left(M_{(n)} \mid m_{\max}\right) = m_{\min} + \left(m_{\max} - m_{\min}\right) - \int_{m_{\min}}^{m_{\max}} F_{M_{(n)}}\left(m \mid m_{\max}\right) dm.$$

If we eliminate the minimum $m_{\min}$ we shall get back the formula (5), but we replace the difference $m_{\max} - m_{\min}$ with the series (11) and integral with the series (12) instead, so we get

$$E\left(M_{(n)} \mid m_{\max}\right) = m_{\min} + \frac{1}{\beta}\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^k}{k} - \frac{1}{\beta}\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^k}{k+n}.$$

To each term it holds

$$\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^k}{k} - \frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^k}{k+n} = n\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^k}{k\left(k+n\right)}$$

so it implies that

$$\beta\left[E\left(M_{(n)} \mid m_{\max}\right) - m_{\min}\right] = n\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^k}{k\left(k+n\right)}. \tag{15}$$

Here the right side is the Kijko-Sellevoll function 2 (KS-2 or f_n^{KS-2}). The relation between KS-1 and KS-2 is

$$\begin{aligned}\beta\left(m_{\max}-m_{\min}\right) &= \sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^k}{k+n} + n\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^k}{k\left(k+n\right)} \\ &= f_n^{KS-1}\left(\beta\left(m_{\max}-m_{\min}\right)\right) + f_n^{KS-2}\left(\beta\left(m_{\max}-m_{\min}\right)\right).\end{aligned} \tag{16}$$

Using this relationship, we can find the finite sum formula for KS-2

$$f_n^{KS-2}\left(\beta\left(m_{\max}-m_{\min}\right)\right) = \frac{\left[\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^n - 1\right]\beta\left(m_{\max}-m_{\min}\right) + \sum_{k=1}^{n}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^k}{k}}{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^n} \tag{17}$$

The expression (17) can be used to calculate numerical values to the function KS-2. Any way as we said before (in the case KS-1), it is recommended to use the numerical solution of series instead of formula (17) because of numerical stability. We could also calculate KS-2 when we know KS-1 by means of relation (16).

The function KS-2 gives more «natural way» the solution of $m_{\max}$ the than the function KS-1 because there is some inverse function such that

$$f^{-1}\left(\beta\left[E\left(M_{(n)} \mid m_{\max}\right)-m_{\min}\right]\right)=\beta\left(m_{\max}-m_{\min}\right).$$

Unfortunately, we cannot say what is the inverse function.

When maximum $m_{\max}$ is considered to be infinity, and taking into account KS-1 function for fixed $n \geq 0$, we have

$$\lim_{m_{\max}\to\infty}\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{k}}{k+n}=\sum_{k=1}^{\infty}\frac{1}{k+n}=\sum_{k=1}^{\infty}\frac{1}{k}-\sum_{k=1}^{n}\frac{1}{k}\to\infty,$$

so the KS-1 diverges because the harmonic series diverges. Hence the KS-1 is an unbounded function.

The case of the KS-2 function is different. We need some results of Psi function (Abramowitz and Stegun, 1972)

$$\psi(1+z)=-\gamma+\sum_{k=1}^{\infty}\frac{z}{k(k+z)}, \qquad z\neq-1,-2,-3,\ldots,$$

$$\psi(1)=-\gamma,\ \psi(n)=-\gamma+\sum_{k=1}^{n-1}\frac{1}{k}, \qquad n\geq 2,$$

where γ is the Euler-Mascheroni constant. When n is zero, the KS-2 function is zero. Let's assume that $n \geq 0$ is fixed then the KS-2 function converges since it is positive term series and

$$\lim_{m_{\max}\to\infty}\sum_{k=1}^{\infty}\frac{n\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{k}}{k(k+n)}=n\sum_{k=1}^{\infty}\frac{1}{k(k+n)}\leq n\sum_{k=1}^{\infty}\frac{1}{k^{2}}<\infty \tag{18}$$

because the last series converges. Moreover, if n is positive integer ($n \geq 1$) then

$$\sum_{k=1}^{\infty}\frac{n}{k(k+n)}=\psi(1+n)+\gamma=\sum_{k=1}^{n}\frac{1}{k}=H_{n}, \tag{19}$$

where H_n is called the harmonic number. Thus the KS-2 is a bounded function for any fixed n.

If we apply the results (18) and (19) to the equation we get

$$E\left(M_{(n)} \mid \infty\right)-m_{\min}=\frac{H_{n}}{\beta}. \tag{20}$$

Because of we assumed that the value β is constant, the right side of (20) is constant for any fixed n. The formula (20) shows a fundamental phenomenon of unbounded distribution function. No matter how we choose the minimum $m_{\min}$, the distance to the expected maximum value is always the same.

Estimator for $m_{\max}$

As we showed there are two different solutions for the estimator $E\left(M_{(n)} \mid m_{\min}\right)$

$$E\left(M_{(n)} \mid m_{\max}\right) = m_{\max} - \frac{1}{\beta} f_n^{KS-1}\left(\beta\left(m_{\max} - m_{\min}\right)\right)$$
$$= m_{\min} + \frac{1}{\beta} f_n^{KS-2}\left(\beta\left(m_{\max} - m_{\min}\right)\right).$$

Also our auxiliary function can (6) be written using the KS-1 or KS-2 function

$$g\left(\mathfrak{M}\right) = \mathfrak{M} - E\left(M_{(n)} \mid m_{\max}\right) - \frac{1}{\beta} f_n^{KS-1}\left(\beta\left(\mathfrak{M} - m_{\min}\right)\right)$$
$$= m_{\min} - E\left(M_{(n)} \mid m_{\max}\right) + \frac{1}{\beta} f_n^{KS-2}\left(\beta\left(\mathfrak{M} - m_{\min}\right)\right).$$

Since they present the same function, they have the same derivative. We get

$$g'\left(\mathfrak{M}\right) = \frac{1}{\beta}\frac{\partial}{\partial \mathfrak{M}} f_n^{KS-2}\left(\beta\left(\mathfrak{M} - m_{\min}\right)\right)$$
$$= n\frac{\exp\left[-\beta\left(\mathfrak{M} - m_{\min}\right)\right]}{1 - \exp\left[-\beta\left(\mathfrak{M} - m_{\min}\right)\right]} f_n^{KS-1}\left(\beta\left(\mathfrak{M} - m_{\min}\right)\right).$$

Now the step of Newton-Raphson method (NRM) has the expression

$$\mathfrak{M}_0 = E\left(M_{(n)} \mid m_{\max}\right),$$
$$\mathfrak{M}_{k+1} = \mathfrak{M}_k - \frac{g\left(\mathfrak{M}_k\right)}{g'\left(\mathfrak{M}_k\right)} = \mathfrak{M}_k - \frac{\mathfrak{M}_k - E\left(M_{(n)} \mid m_{\max}\right) - \frac{1}{\beta} f_n^{KS-1}\left(\beta\left(\mathfrak{M}_k - m_{\min}\right)\right)}{n\frac{\exp\left[-\beta\left(\mathfrak{M}_k - m_{\min}\right)\right]}{1 - \exp\left[-\beta\left(\mathfrak{M}_k - m_{\min}\right)\right]} f_n^{KS-1}\left(\beta\left(\mathfrak{M}_k - m_{\min}\right)\right)}$$

$$= \mathfrak{M}_k + \frac{\exp\left[\beta\left(\mathfrak{M}_k - m_{\min}\right)\right] - 1}{n\beta}\left\{1 - \frac{\beta\left[\mathfrak{M}_k - E\left(M_{(n)} \mid m_{\max}\right)\right]}{f_n^{KS-1}\left(\beta\left(\mathfrak{M}_k - m_{\min}\right)\right)}\right\}.$$

The second term in the difference between brackets, measures the distance to the exact solution when it is different than one. This step is a Pisarenko estimator (Pisarenko et al., 1996) or a Tate-Pisarenko estimator (21) (Kijko and Graham, 1998), at $\mathfrak{M}_0 = E\left(M_{(n)} \mid m_{\max}\right)$ (using the estimator of the maximum observed magnitude $m_{(n)}$)

$$\hat{m}_{\max} = m_{(n)} + \frac{1}{n\beta} \frac{1-\exp\left[-\beta\left(m_{(n)} - m_{\min}\right)\right]}{\exp\left[-\beta\left(m_{(n)} - m_{\min}\right)\right]}. \tag{21}$$

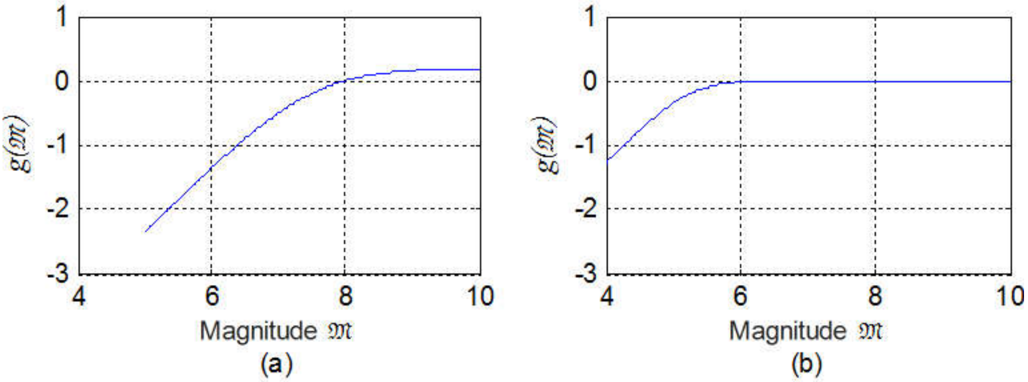

Figure 1: Function with $g(\mathfrak{M})$ whith different set of parameters.

We show in figure 1 how the function g (in ordinates) varies with magnitude (in abscissas); it looks like Γ. The figure 1a has been drawn with parameters $b=1, m_{\max}=8, m_{\min}=5$ and $n=200$, and the figure 1b with $b=2,\ m_{\max}=9.5,\ m_{\min}=4$ and $n=200$. As we can see in case 1b the derivative is almost zero and NRM cannot solve it using double-precision arithmetic. In simulations the NRM can find the solution up to $b\left(m_{\max}-m_{\min}\right)=7$ with exact expected value, but it is numerically stable up to $b\left(m_{\max}-m_{\min}\right)=4.5$.

The artificial catalogue can be generated by using inverse function of (3)

$$m_i = m_{\min} - \frac{1}{\beta}\log\left[1-\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)u_i\right], \tag{22}$$

where $u_i \in U\left(\left[0,1\right]\right)$ is uniformly distributed random variable between zero and one. From this artificial catalogue we choose the maximum observed value $m_{(n):k} = \max\left\{m_1,\ldots,m_n\right\},\ m_i \in C_{n:k}$.

In order to get some estimator of the expected value $E\left(M_{(n)} \mid m_{\max}\right)$, we could have

$$\bar{\hat{m}}_{\max} = m_{\min} + \frac{1}{\beta N}\sum_{k=1}^{N}\left(f_n^{KS-2}\right)^{-1}\left(\beta\left[m_{(n):k} - m_{\min}\right]\right),$$

where $\bar{\hat{m}}_{\max}$ is the mean value of the estimators of $m_{\max}$. In this case we use a single maximum event to estimate the expected value of maximum to each catalogue and finally we calculate the mean of maximums.

On the other hand, we can calculate first the mean value of maximums of all catalogues, such like

$$\bar{m}_{(n)} = \frac{1}{N}\sum_{k=1}^{N} m_{(n):k} \tag{23}$$

and consider the expression(23) 0as the estimator of the expected value of the maximum $E\left(M_{(n)} \mid m_{\max}\right)$. Thus, we obtain the estimator of $m_{\max}$ using the mean value of maximums of all catalogues

$$\hat{\bar{m}}_{\max} = m_{\min} + \frac{1}{\beta}\left(f_n^{KS-2}\right)^{-1}\left(\beta\left[\frac{1}{N}\sum_{k=1}^{N} m_{(n):k} - m_{\min}\right]\right).$$

The main problem is that the Kijko-Sellevoll functions f_n^{KS-1} and f_n^{KS-2} map the interval $[m_{\min}, m_{\min} + H_n/\beta] \subseteq [m_{\min}, m_{\max}]$ into $[m_{\min}, \infty]$. So if the estimator is greater than the value $m_{\min} + H_n/\beta$ then the solution is beyond infinity. For example let $b=1,\ m_{\max}=8,\ m_{\min}=5$ and $n=1$, then the estimator of the maximum observed magnitude must lie in the interval $[5, 5.4342]$. Due to the maximum magnitude is 8, it is clear that in this case, when $n=1$, often there will be the event greater than 5.4342, and consequently, with mean value superior to this upper limit.

Examples and simulations

In figure 2 we show the problem of simulation for $b=1,\ m_{\max}=8,\ m_{\min}=5$. To each catalogue size (that is to say n) it was generated 1000 artificial catalogues. The figure 2b shows how many of them were accepted i.e. the maximum observed value of catalogue was smaller than $m_{(n):k} < m_{\min} + H_n/\beta$. We calculate the estimator of the maximum for each of the accepted catalogues and afterwards the mean values of those maximums of the catalogues of each size n. The results are plotted in the figure 2a. In the figure 2c we applied the formula (23). For each size n, all the 1000 catalogues are used to estimate the expected value of maximum and by using this estimator (only one value) we calculate the value of the estimator of the maximum $\hat{\bar{m}}_{\max}$. The figure 2d shows how many attempts were necessary to make to get a mean value of 1000 catalogues, which fulfills $\bar{m}_{(n)} < m_{\min} + H_n/\beta$. We can see from 2c and 2d that for catalogues of size greater than 10, the method behaves quite stable.

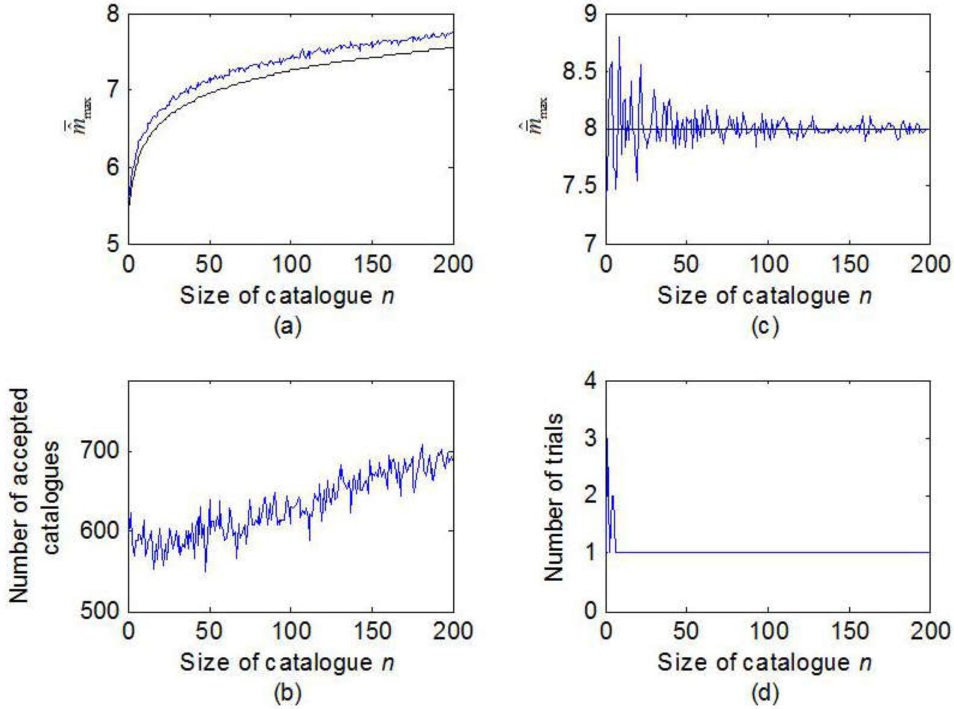

Figure 2: The case $1(8-5)$ and sample size1000 simulation.

This result is quite expected. In the first case (illustrated by figure 2a) we are modeling the expected value of the maximum with the mean value of only one event so the variance is quite big. For small size catalogues we have 55 to 60% of acceptance of the requirements. The situation is not so much better even though the catalogue size is 200, since still almost 30 % of maximums are rejected. In the second case (figure 2c) all the 1000 maximums are used to calculate the mean value estimator. Its variance decreases as the square root of size of sample, so the variance comes 30 times smaller than in the first case.

Of course we could have enough big catalogue size that all events are included into the acceptable interval, so the maximum $m_{\max}$ (unknown) also does; to the number of events n, holds $m_{\max} \leq m_{\min} + H_n/\beta$. For example if $m_{\min} = 5$ and $b = 1$, then n is 56, 561, 5615 and 56146 for the maximum values 7, 8, 9 and10, respectively. With the real catalogues this is not possible because we should wait years or hundreds years to gather more data.

Another thing what we can do is to increase the minimum value; for example if $m_{\min} = 6$, then n is,6, 56, 561 and 5615 for the maximum values 7,8,9 and 10, respectively.

Next we consider less number of catalogues. The figure 3 shows the result of considering $m_{\min} = 6$, and $b = 1$ and 100 simulated catalogues; the figure 3b shows that we could use a single catalogue when n is more than about 50. Similarly figure 4 displays the case when minimum value is 7 and only 10 simulated catalogues are taken into account; the figure 4b shows that almost always it is possible to use one single catalogue if the distance to maximum is about one magnitude unit. Those three figures show that the situation of the single catalogue comes better as the distance between maximum and minimum comes smaller. Also 2c, 3c and 4c shows that it is possible to analyze the behavior of KS function in extreme cases like small catalogue sizes and/or big value of $b(m_{max} - m_{min})$, which can be of extremely importance in zones with few

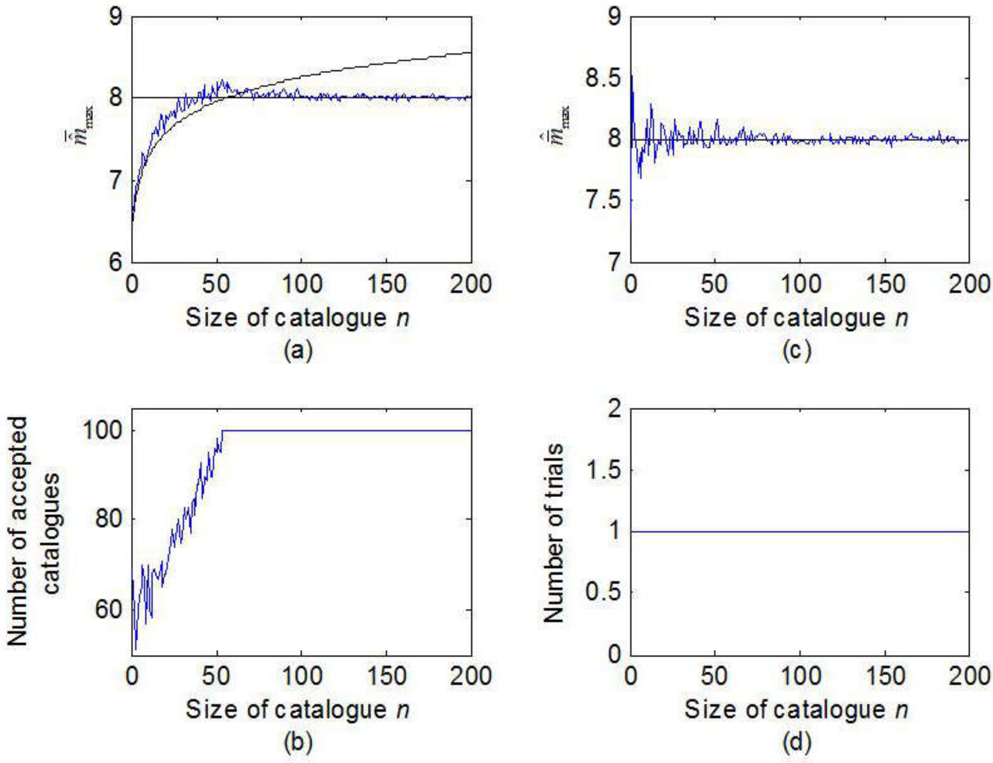

Figure 3: The case 1 (8 - 6) and sample 100 size simulation.

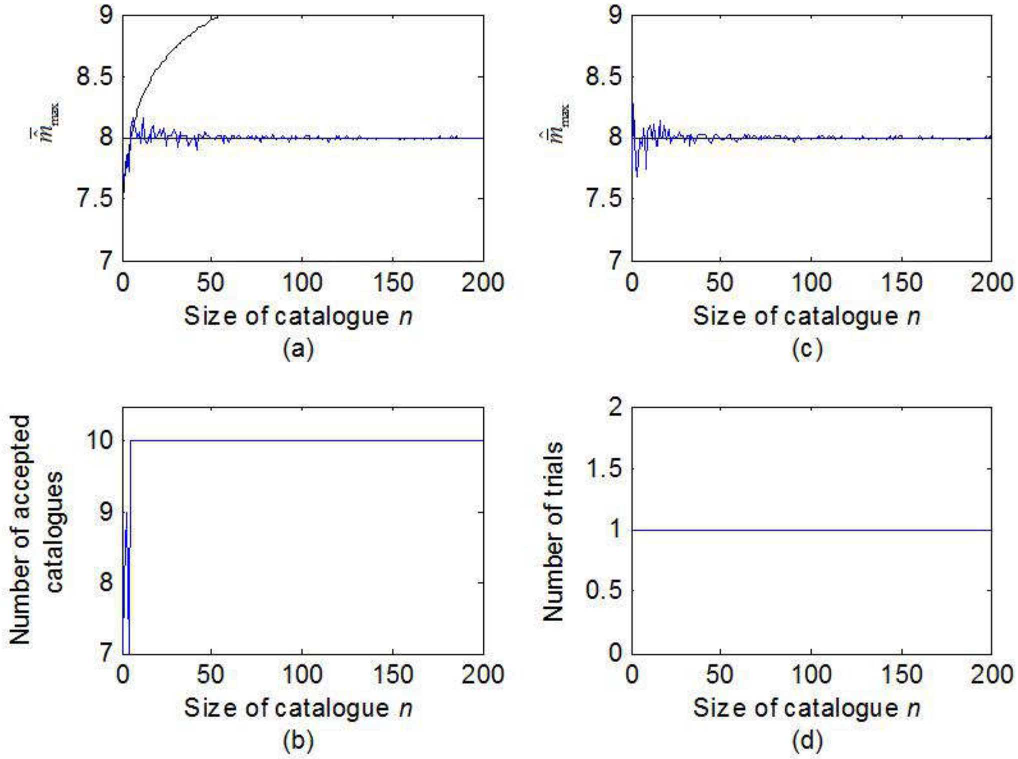

Figure 4: The case 1 (8 - 7) and sample size10 simulation.

Concluding remarks

In this work we report a method to solve exactly Kijko-Sellevoll formula to calculate $b(m_{max} - m_{min})$, considering throughout the text an exact value for b. In fact, both parameters are closely related when we are dealing with seismic hazard assessment. In a following paper we shall show that the estimators of Cosentino et al. (1976, 1977), Page (1968) and Aki-Utsu (1965) are related with the KS-2.

The series resulted to be not only a tool to solve an equation but they also let us to build a rich theory. They give numerically stable method to manage wider range of magnitudes and size of catalogues. The cost of this is to have more complicated calculus (but not so much slower). The exact solution of KS estimator does not only mean the solution of the problem without approximations, besides it makes possible a numerically «exact» solution and the improvement of the computer performance. At least our work gives an alternative viewpoint to see and analyze other similar methods.

We used a fixed β to all catalogues. In fact this is not realistic since always the β -value must be estimated to each catalogue and that estimator changes from one catalogue to another. That could make the method more «soft,» but still there will be failed catalogues. We shall go insight into this topic in another report (Part II). As we showed, the way to avoid the problem in the case of failing catalogues is to put the minimum $m_{\min}$ bigger even the number of the events of the catalogue will come smaller. As we could see from the figures, when the minimum $m_{\min}$ is closer to the maximum $m_{\max}$, we need less data to get answers. The variance of the estimator of the maximum comes smaller as the difference $m_{\max} - m_{\min}$ comes smaller, even we have used fewer events in catalogues. Kijko (2004) showed this fact empirically in his simulations.

Acknowledgments

We are grateful to J. Seppänen for his help to get a valuable paper necessary to complete this work.

Appendix A

To the readers, who are not familiar with KS estimator, we shall give it shortly (the reader can find more details in the works of Kijko and Graham (1998) and Kijko (2004)).

First we remark that Cramér's (1961) approximation

$$\left[f(x)\right]^n \approx \exp\left\{-n\left[1-f(x)\right]\right\}$$

can be derived (using) as

$$\begin{aligned}
0 \le \left[f(x)\right]^n &= \exp\left\{n \log\left(f(x)\right)\right\} \\
&= \exp\left\{-n\left[-\log\left(1-\left[1-f(x)\right]\right)\right]\right\} \\
&= \exp\left\{-n\sum_{k=1}^{\infty}\frac{\left[1-f(x)\right]^k}{k}\right\} \\
&= \exp\left\{-n\left[1-f(x)\right]-n\sum_{k=2}^{\infty}\frac{\left[1-f(x)\right]^k}{k}\right\} \\
&= \exp\left\{-n\left[1-f(x)\right]\right\}\exp\left\{-n\sum_{k=2}^{\infty}\frac{\left[1-f(x)\right]^k}{k}\right\} \\
&\le \exp\left\{-n\left[1-f(x)\right]\right\}.
\end{aligned}$$

We see that this approximation comes from the linearization of logarithm and equality holds when $n=0$ or $f(x)=1$. This inequality also shows that Cramér's approximation overestimate the original CDF.

Applying Cramér's approximation to integral (5) we have

$$\begin{aligned}
\Delta &\approx \int_{m_{\min}}^{m_{\max}} \exp\left\{-n\left[1-\frac{1-\exp\left[-\beta\left(m-m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]}\right]\right\}dm \\
&= \int_{m_{\min}}^{m_{\max}} \exp\left\{-\frac{n\exp\left(-\beta\left(m-m_{\min}\right)\right)}{1-\exp\left(-\beta\left(m_{\max}-m_{\min}\right)\right)}+\frac{n\exp\left(-\beta\left(m_{\max}-m_{\min}\right)\right)}{1-\exp\left(-\beta\left(m_{\max}-m_{\min}\right)\right)}\right\}dm \\
&= \frac{\displaystyle\int_{m_{\min}}^{m_{\max}} \exp\left\{-\frac{n\exp\left(-\beta\left(m-m_{\min}\right)\right)}{1-\exp\left(-\beta\left(m_{\max}-m_{\min}\right)\right)}\right\}dm}{\exp\left\{-\frac{n\exp\left(-\beta\left(m_{\max}-m_{\min}\right)\right)}{1-\exp\left(-\beta\left(m_{\max}-m_{\min}\right)\right)}\right\}} \\
&=\because
\end{aligned}$$

Setting now

$$\zeta = \frac{n\exp\left(-\beta\left(m-m_{\min}\right)\right)}{1-\exp\left(-\beta\left(m_{\max}-m_{\min}\right)\right)} \quad \Rightarrow \quad \frac{d\zeta}{dm} = -\beta\zeta$$

$$n_1\left(m_{\max}\right) = \frac{n}{1-\exp\left(-\beta\left(m_{\max}-m_{\min}\right)\right)}$$

$$n_2\left(m_{\max}\right) = n_1\left(m_{\max}\right)\exp\left(-\beta\left(m_{\max}-m_{\min}\right)\right)$$

we can write

$$\because = \frac{-\int_{n_1(m_{\max})}^{n_2(m_{\max})}\exp\left(-\zeta\right)\frac{d\zeta}{\beta\zeta}}{\exp\left(-n_2\left(m_{\max}\right)\right)} = \frac{\int_{n_2(m_{\max})}^{\infty}\frac{\exp\left(-\zeta\right)}{\zeta}d\zeta - \int_{n_1(m_{\max})}^{\infty}\frac{\exp\left(-\zeta\right)}{\zeta}d\zeta}{\beta\exp\left(-n_2\left(m_{\max}\right)\right)}.$$

Note that $n_1\left(\mathfrak{M}\right)-n_2\left(\mathfrak{M}\right)=n$ for all $\mathfrak{M}$. We can approximate the exponential integral by

$$E_1\left(z\right) = \int_z^{\infty}\frac{\exp\left(-\zeta\right)}{\zeta}d\zeta = \frac{z^2+a_1z+a_2}{z\left(z^2+b_1z+b_2\right)}\exp\left(-z\right).$$

$$a_1 = 2.334733 \quad a_2 = 0.250621$$

$$b_1 = 3.330657 \quad b_2 = 1.681534$$

KS estimator is given now

$$\mathfrak{M} = m_{(n)} + \frac{E_1\left(n_2\left(\mathfrak{M}\right)\right)-E_1\left(n_1\left(\mathfrak{M}\right)\right)}{\beta\exp\left(-n_2\left(\mathfrak{M}\right)\right)}. \tag{24}$$

This also can be expressed as

$$\hat{m}_{\max} = m_{(n)} + \frac{E_1\left(n_2\left(m_{(n)}\right)\right)-E_1\left(n_1\left(m_{(n)}\right)\right)}{\beta\exp\left(-n_2\left(m_{(n)}\right)\right)}. \tag{25}$$

Kijko (2004) remarked that this estimator (25) can be used when $\left(m_{\max}-m_{\min}\right) < 2$ and $n > 100$.

Appendix B

In this section we discuss the numerical solution of series. The exact solution of the integral is the «core» of the Newton-Raphson method. For example with $\beta(m_{max} - m_{min}) = 1$ and $n = 400$ the solution (9) comes unstable. This is clear because we showed that the numerator is a tail of logarithm function(so the numerator is small as is big) and, if at the same time $1 - \exp(-\beta(m_{max} - m_{min})) \ll 1$ the denominator is very small, yielding equation (9)to instability.

Because of all the series of the Kijko-Sellevoll functions are nonnegative terms series, they have not a similar numerical instability. Because of $\{a_k\}$ is nonnegative sequence, we could calculate the series $S = \sum_{k=0}^{\infty} a_k$ as a partial sum $S_{n_\varepsilon} = \sum_{k=0}^{n_\varepsilon} a_k$, where n_ε is some integer such as $S_{n_\varepsilon} = S_n$ when $n \geq n_\varepsilon$.

The idea sounds quite simple, but the «coin has also other side». The Kijko-Sellevoll function KS-1 belongs to the family of Lerch transcendent function (or shortly Lerch Phi) (Lerch, 1887; Erdélyi et al, 1953))

$$\Phi(z, s, \alpha) = \sum_{k=0}^{\infty} \frac{z^k}{(k+\alpha)^s}$$

and they both (KS-1 and KS-2) are close to logarithm function (we showed that KS-1 is scaled tail of logarithm function). This means that the convergence of the series is slowly, so we need to use some acceleration algorithm to solve the value of the series. For example in the extreme case when $b(m_{max} - m_{min}) = 16$ we might need approximately 10^{19} terms to calculate the value using the direct sum of the series, but with accelerator (discussed below) we need only about 10^3 terms from the series.

The following algorithm we present here, bases on the paper of Cohen et al. (2000) and their algorithm 2_A. We tried also the algorithm 2_B, but we did not get better results than those reached by the former expression, so we adopted it, even Cohen et al. (2000) recommended method 2_B. This algorithm presented below can be applied also to other alternative or nonnegative series different than KS functions. Because of the numerical solution of the integral is so important tool to the analysis of the exact solution of Kijko-Sellevoll estimator, we give also the open source code in MATLAB.

Having a series

$$S = \sum_{k=0}^{\infty} (-1)^k a_k,$$

where a_k is well-behaved function which goes to zero as $k \to \infty$ (sequence of the series), we want to find coefficients $c_{n,k}/d_n$ such that the sequence

$$S_n = \sum_{k=0}^{n-1} \frac{c_{n,k}}{d_n} a_k \qquad (26)$$

converges quickly to zero i.e. $|S - S_n| < C^{-n}$ to some constant C. Cohen et al (2000) showed that algorithm 2_A has convergence factor 7.89 for a large class of sequences $\{a_k\}$ and 17.93 for a small class of sequences. To the algorithm 2_B was reported factor 9.56 for a large class and 14.41

for a small class. From our experiments, we can say that our sequences converge approximately with factor 10 which means a one correct decimal in the sequence (26) for each term in the sum.

The algorithm of Cohen et al. bases on Chebyshev polynomials. It is set $P_n\left(\sin^2 t\right)=\cos 2nt$ so that $P_n(x)=T_n(1-2x)$, where $T_n(x)$ is the ordinary Chebyshev polynomial. Clearly $P_0(x)=1$. Any way this can be any arbitrary constant. Now the sequence of polynomials can be given as

$$P_n(x)=\sum_{m=0}^{n}(-1)^m\frac{n}{n+m}\binom{n+m}{2m}2^{2m}x^m.$$

Using these polynomials, it can be defined a new family of polynomials

$$P_n^{(m)}(x)=\sum_{r=0}^{m}(-1)^r\binom{m}{r}(n-2r)^{m+1}P_{n-2r}(x).$$

Here we can see that when $n-2r=0$ (case when we have P_0) the factor of the polynomial is zero, so P_0 can be any constant. The suggested sequence of polynomials are defined as

$$A_n(x)=\frac{P_n^{(n-1)}(x)}{n!\,2^{n-1}}.$$

The normalization factor has been chosen to fulfill $A_n(0)=1$. Any way it is an arbitrary factor and we could choose directly $A_n(x)=P_n^{(n-1)}(x)$. The algorithm of Cohen et al (2000) is

Let $A_n(x)=\sum_{k=0}^{n}b_k x^k$
$d=A_n(-1);\ c_{n,0}=-d;\ s=0$
For $k=0$ up to $k=n-1$, repeat:
  $c_{n,k+1}=-b_k-c_{n,k};\quad s=s+c_{n,k+1}\cdot a_k;$
Output: s/d

This algorithm has been written to evaluate the factors «on the fly.» In our case the degree of the polynomial is fixed because we do not want to use time to recalculate the factors in each time when the program is called. These factors are universals and their values depend only on the degree of the polynomial. From the algorithm we can see that the factors are

$$c_{n,0}=-\sum_{m=0}^{n}|b_m|,\ c_{n,1}=\sum_{m=1}^{n}|b_m|,\ c_{n,2}=-\sum_{m=2}^{n}|b_m|,\ \ldots$$
$$c_{n,k}=(-1)^{k+1}\sum_{m=k}^{n}|b_m|,$$

so $d=-c_{n,0}$. Normalized factors result equal to $\tilde{c}_{n,k}=c_{n,k}/d=-c_{n,k}/c_{n,0}$, which in the case of $n=18$, give the next values:

$$
\begin{array}{lll}
\tilde{c}_{18,1} = 0.999999999999999245 & \tilde{c}_{18,7} = 0.999816879032895474 & \tilde{c}_{18,13} = 0.580476889354509827 \\
\tilde{c}_{18,2} = -0.999999999998277922 & \tilde{c}_{18,8} = -0.998595555793887371 & \tilde{c}_{18,14} = -0.356402292890562931 \\
\tilde{c}_{18,3} = 0.999999999610062291 & \tilde{c}_{18,9} = 0.992342806734044361 & \tilde{c}_{18,15} = 0.168952372776841569 \\
\tilde{c}_{18,4} = -0.999999972513434920 & \tilde{c}_{18,10} = -0.969204555863406323 & \tilde{c}_{18,16} = -0.057152687879365596 \\
\tilde{c}_{18,5} = 0.999999109387382570 & \tilde{c}_{18,11} = 0.906022499505876388 & \tilde{c}_{18,17} = 0.012162740075262281 \\
\tilde{c}_{18,6} = -0.999983872560606631 & \tilde{c}_{18,12} = -0.777218084969806462 & \tilde{c}_{18,18} = -0.001216274007526228
\end{array}
$$

Pay attention that in the MATLAB code the factors are not normalized.

We can also see from the algorithm that the partial sum is now

$$S = -\frac{1}{c_0}\sum_{k=0}^{n-1} c_{n,k+1} a_k = \sum_{k=0}^{n-1} \tilde{c}_{n,k+1} a_k \,.$$

The algorithm above is to the alternative series. The nonnegative series can be solved by means of the trick of Van Wijngaarden (Press et al., 1992)

$$\sum_{k=1}^{\infty} a_k = \sum_{m=1}^{\infty} (-1)^m b_m \quad \text{with } b_m = \sum_{k=0}^{\infty} 2^k a_{2^k m} \,.$$

```
function S = KS(ftype,x,n)
%
% Input:
%  ftype = 1: Kijko-Sellevoll function 1
%          2: Kijko-Sellevoll function 2
%          3: A special series for a variance
%   x     = beta*(mmax-mmin)  (scalar or vector)
%   n     = number of events  (scalar or vector)
%

%  Written by Mika Haarala Orosco, Acrenet Oy
%    (21.1.2015 - ver. 14.06.2016)
%
%  Reference:
%    Cohen, H., F. Rodriguez Villegas, and D. Zagier (2000). Convergence
```

```
%    Cohen, H., F. Rodriguez Villegas, and D. Zagier (2000). Convergence
%    acceleration of alternating series, Exper. Math. 9, 3-12.

miter = 10000;
if ~(ftype == 1 || ftype == 2 || ftype == 3), error('Ftype must be 1, 2 or 3.'),
end
if ~(all(size(x) == size(n)) || numel(x) == 1 || numel(n) == 1)
   error('Inputs must be vectors or scalars.')
end

S = NaN( max( size(x), size(n) ) );
z = 1 - exp(-x(:));
n = n(:);

if ftype == 1
    I =  z == 1;
    S(I) = Inf;
else
    I = false( size(z) );
end

I = ~(z < 0 | I | n < 0);
if ~isscalar(z)
    z = z(I);
end
if ~isscalar(n)
    n = n(I);
end

Sn = zeros( size(z) );
So = -1;

if any(z > 0.35)
```

```
% Accelerated sum

f=[1.437775728963973375
  -1.437775728961498499
   1.437775728403331487
  -1.437775689444458316
   1.437774448462769210
  -1.437752541323044337
   1.437512442082007163
  -1.435756453171741645
   1.426766402334197056
  -1.393498786821714133
   1.302657159684823614
  -1.117465298681447728
   0.834595582738420712
  -0.512426566465161050
   0.242915620929416540
  -0.082172747478005376
   0.017487292477909572
  -0.001748729247790957];

for k = 1:length(f)
    fi = 1;
    for i = 1:miter
        switch ftype
            case 1
               Sn = Sn + f(k)*(fi .* z.^(fi*k)./(fi*k + n));
            case 2
              Sn = Sn + f(k)*(fi .* n .* z.^(fi*k)./(fi*k * (fi*k + n)));
            case 3
               Sk = zeros(size(n));
               fik= fi*k;
```

```
                for j = 1:length(n)
                    if fik < 3000
                        Sk(j) = sum( 1./( n(j)+1 : n(j)+fik-1 ) );
                    else
                        Sk(j) = Hn( n(j), n(j)+fik-1 );
                    end
                end
                Sn = Sn + f(k)*(fi * 2*n .* Sk .* z.^(fik + 1)./ ...
                            ((2*n + fik) .* (fik + 1 + n)) );
            end
            if all(Sn == So), break, end
            So = Sn;
            fi = fi*2;
        end
    end
    Sn = Sn/1.437775728963974460;

else
    %  Direct sum

    for k=1:miter
        switch ftype
            case 1
                Sn = Sn + z.^k ./ (k + n);
            case 2
                Sn = Sn + n .* z.^k ./ (k *(k + n));
            case 3
                Sk = zeros(size(n));
                for j = 1:length(n)
                    Sk(j) = sum( 1./( n(j)+1 : n(j)+k-1 ) );
                end
                Sn = Sn + 2*n .* Sk .* z.^(k+1) ./ ( (2*n + k) .* (k + n) );
        end
```

```
        if all(Sn == So), break, end
        So = Sn;
    end

end

S(I) = Sn;




function y = Hn(k1,k2)
%
%  This function calculates the subtraction of Harmonic numbers:
%    y = sum(1./(k1+1:k2)) = sum(1./(1:k2)) - sum(1./(1:k1))
%
%   Reference:
%     Villarino, M.B. 2008, Ramanujan's Harmonic Number Expansion
%     into Negative Powers, J. Inequal. Pure and Appl. Math., 9(3),
%     Art. 89, 12 pp.
%

euler = 0.57721566490153286;

if k1 < 10
    D  = ceil(10-k1);
    m  = (k1 + D) * (k1 + D + 1) / 2;
    y1 = euler + log(2*m)/2 + 1/(12*m) - 1/(120*m^2)...
          + 1/(630*m^3) - 1/(1680*m^4) + 1/(2310*m^5)...
          - 191/(360360*m^6) - sum( 1./ (k1+D:-1:k1+1) );
else
    m  = k1 * (k1+1) / 2;
    y1 = euler + log(2*m)/2 + 1/(12*m) - 1/(120*m^2)...
          + 1/(630*m^3) - 1/(1680*m^4) + 1/(2310*m^5)...
```

```
end

m  = k2 * (k2+1) / 2;
y2 = euler + log(2*m)/2 + 1/(12*m) - 1/(120*m^2)...
      + 1/(630*m^3) - 1/(1680*m^4) + 1/(2310*m^5)...
      - 191/(360360*m^6);

y = y2 - y1;
```

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