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Analysis of Gutenberg-Richter b-value and m_{max}. Part III: Non-positive Gutenberg-Richter b-value

Análisis del parámetro b y m_{max} del Modelo de Gutenberg-Richter. Parte III: valor no positivo del parámetro b de Gutenberg-Richter

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Probabilidad – Ingeniería Sísmica / artículo científico

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Abstract

When we analyzed the Gutenberg-Richter distribution function in our earlier works, we assumed that the b-value is positive. Using generalized estimators, we found that in some cases the b-value can be also negative. This paper gives a theoretical background for the negative b-value. We also expand the KS functions on the interval $-\infty < \beta(m_{\max} - m_{\min}) \leq -\log(2)$.

Keywords: Gutenberg-Richter distribution function, Gutenberg-Richter b-value, Kijko-Sellevoll functions.

Resumen

Cuando en trabajos anteriores analizamos la función de distribución de Gutenberg-Richter, asumimos un valor positive para el parámetro b. Con el uso de distintos estimadores, encontramos que este parámetro puede tomar también valores negativos. En este artículo se establece un marco teórico para el caso de valor negativo de b y demostraremos la expansión de la función Kijko-Sellevoll (KS) al intervalo $-\infty < \beta(m_{\max} - m_{\min}) \leq -\log(2)$.

Palabras clave: Función de distribución de Gutenberg-Richter, parámetro b de Gutenberg-Richter, funciones de Kijko-Sellevoll

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1. Introduction

In earlier works (Haarala and Orosco, 2016a, 2016b, 2018) we have studied the double truncated exponential probability density function (PDF), or called also as the Gutenberg-Richter probability density function (GR),

$$f(m) = \frac{\beta \exp[-\beta(m - m_{\min})]}{1 - \exp[-\beta(m_{\max} - m_{\min})]} \tag{1}$$

(where $\beta = b\log(10)$) assuming that the β is always positive. Even if we generate the data with positive β-value, the generalized estimators can give negative β-values. We gave an example in a previous work (Haarala and Orosco, 2016b) where we could not get the values with the generalized Page estimator with our program. We set the estimates $\hat{\beta}_{GP}^{(n)}$ to zero (as we can see the Page estimates at the points for $n \geq 188$ in the Figure 1) without knowing that they are negative values.

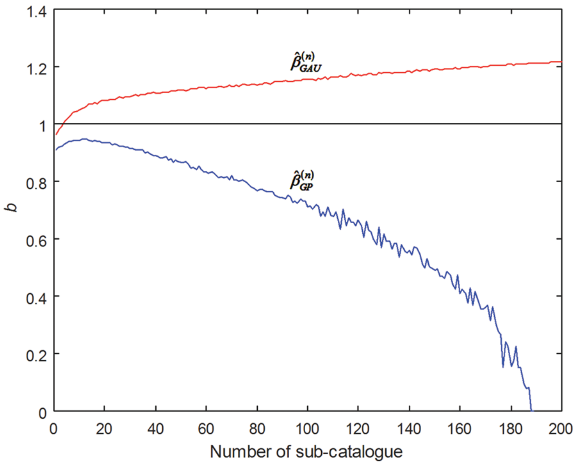

Figure 1. Example of Generalized Aki-Utsu (GAU) and Page (GP) estimators (Haarala and Orosco, 2016b)

A reason for this «failure» was our assumption that the b-value is always positive. Another reason was the discontinuity of the PDF (1) at $b = 0$. When we proved more general and simple results for the Kijko-Sellevoll (KS) functions, we found their real convergence interval $-\log(2) \leq \beta(m_{\max} - m_{\min}) < \infty$ even though we gave the proof only to the positive interval $0 \leq \beta(m_{\max} - m_{\min}) < \infty$ (Haarala and Orosco, 2018). In this article we focused to the negative part of the Kijko-Sellevoll (KS) functions, which will yield the solutions for the interval $-\infty < \beta(m_{\max} - m_{\min}) < 0$.

2. Generalization of Gutenberg-Richter distribution function

Let's consider the distribution function (1) which has the cumulative distribution function (CDF)

$$F_M(m)=\begin{cases}0, & \text{for } m<m_{\min},\\ \dfrac{1-\exp\left[-\beta\left(m-m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]}, & \text{for } m_{\min}\le m<m_{\max},\\ 1, & \text{for } m_{\max}\le m,\end{cases} \tag{2}$$

where $-\infty\le\beta\le\infty$ and $-\infty\le m_{\min}<m_{\max}\le\infty$. The difference $m_{\max}-m_{\min}$ is positive always, so the factor $\beta\left(m_{\max}-m_{\min}\right)$ can be negative only if $\beta<0$ (i.e. $b<0$). We can see that CDF (2) has a discontinuity at $\beta=0$, where both the numerator and the denominator are zero.

If β is negative, it still holds that $f(m)\ge 0$ for all $m\in\left[m_{\min},m_{\max}\right]$ in the PDF (1) because of $\beta\exp\left[-\beta\left(m-m_{\min}\right)\right]<0$ and $1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]<0$. The CDF (2) holds also, since $F_M(m)\ge 0$ for all $m\in\left[m_{\min},m_{\max}\right]$ because of both the nominator and the denominator are negative at the same time. It is not difficult to see from (2) that

$$F_M(-\infty)=F_M\left(m_{\min}\right)=0,$$
$$F_M(\infty)=F_M\left(m_{\max}\right)=1$$

and F_M is a non-decreasing right continuous function.

If $\beta=0$, the limit of the PDF of GR distribution function can be gotten as

$$\begin{aligned}f(m)&=\frac{\beta\exp\left[-\beta\left(m-m_{\min}\right)\right]}{1-\sum_{k=0}^{\infty}\dfrac{\left[-\beta\left(m_{\max}-m_{\min}\right)\right]^k}{k!}}\\ &=\frac{\exp\left[-\beta\left(m-m_{\min}\right)\right]}{\left(m_{\max}-m_{\min}\right)\left\{1+\sum_{k=2}^{\infty}\dfrac{\left[-\beta\left(m_{\max}-m_{\min}\right)\right]^{k-1}}{k!}\right\}}\\ &\to\frac{1}{m_{\max}-m_{\min}},\end{aligned}$$

when $\beta\to 0$. This is a Uniform Distribution function. It's CDF is well known, but we can get it also by

$$\frac{1-\exp\left[-\beta\left(m-m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]}=\frac{1-\sum_{k=0}^{\infty}\frac{\left[-\beta\left(m-m_{\min}\right)\right]^{k}}{k!}}{1-\sum_{k=0}^{\infty}\frac{\left[-\beta\left(m_{\max}-m_{\min}\right)\right]^{k}}{k!}}$$

$$=\frac{\left(m-m_{\min}\right)\left\{1+\sum_{k=2}^{\infty}\frac{\left[-\beta\left(m-m_{\min}\right)\right]^{k-1}}{k!}\right\}}{\left(m_{\max}-m_{\min}\right)\left\{1+\sum_{k=2}^{\infty}\frac{\left[-\beta\left(m_{\max}-m_{\min}\right)\right]^{k-1}}{k!}\right\}}$$

$$\rightarrow\frac{m-m_{\min}}{m_{\max}-m_{\min}}.$$

Now we can complete the definition of the General Gutenberg-Richter (GGR) distribution function. The PDF is defined as

$$f(m)=\begin{cases}\dfrac{\beta\exp\left[-\beta\left(m-m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]}, & \text{for } m_{\min}\le m\le m_{\max}\wedge\beta\neq 0,\\ \dfrac{1}{m_{\max}-m_{\min}}, & \text{for } m_{\min}\le m\le m_{\max}\wedge\beta=0,\\ 0, & \text{for } m\notin\left[m_{\min},m_{\max}\right]\end{cases}$$

with CDF

$$F_M(m)=\begin{cases}0, & \text{for } m<m_{\min},\\ \dfrac{1-\exp\left[-\beta\left(m-m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]}, & \text{for } m_{\min}\le m<m_{\max}\wedge\beta\neq 0,\\ \dfrac{m-m_{\min}}{m_{\max}-m_{\min}} & \text{for } m_{\min}\le m<m_{\max}\wedge\beta=0,\\ 1, & \text{for } m\ge m_{\max},\end{cases}\tag{3}$$

where $-\infty\le\beta\le\infty$ and $-\infty\le m_{\min}<m_{\max}\le\infty$. We will show later that β is always bounded for practical applications. That is to say, we could assume directly $-\infty<\beta<\infty$.

Figures 2a and 2b illustrate this process with parameters $b=0$, $m_{\min}=5$, $n=1$ using different values for parameter $m_{\max}$. In the figures 2c and 2d, the parameters are $b=-0.1$, $m_{\min}=5$, $n=1$ with different values of parameter $m_{\max}$. We can see from these figures how the probability decreases in small values and concentrate to $m_{\max}$ when $m_{\max}\rightarrow\infty$. Even in the case of the Uniform distribution function, the events in the interval $5\le m\le 10$ become so rare that it is more probable to get a lot of huge values than small values when $m_{\max}\rightarrow\infty$. This fact made us suspect that $m_{\max}$ is bounded, when b is negative.

The negative β has an opposite behavior than the positive one. While the positive β concentrates the events close to the minimum limit, the negative β concentrates the events close to the maximum limit. The figure 3, which was generated using $m_{\max}=8$, $m_{min}=5$, and the b-values 1 and -1 (both figures have 100 events), illustrate this situation.

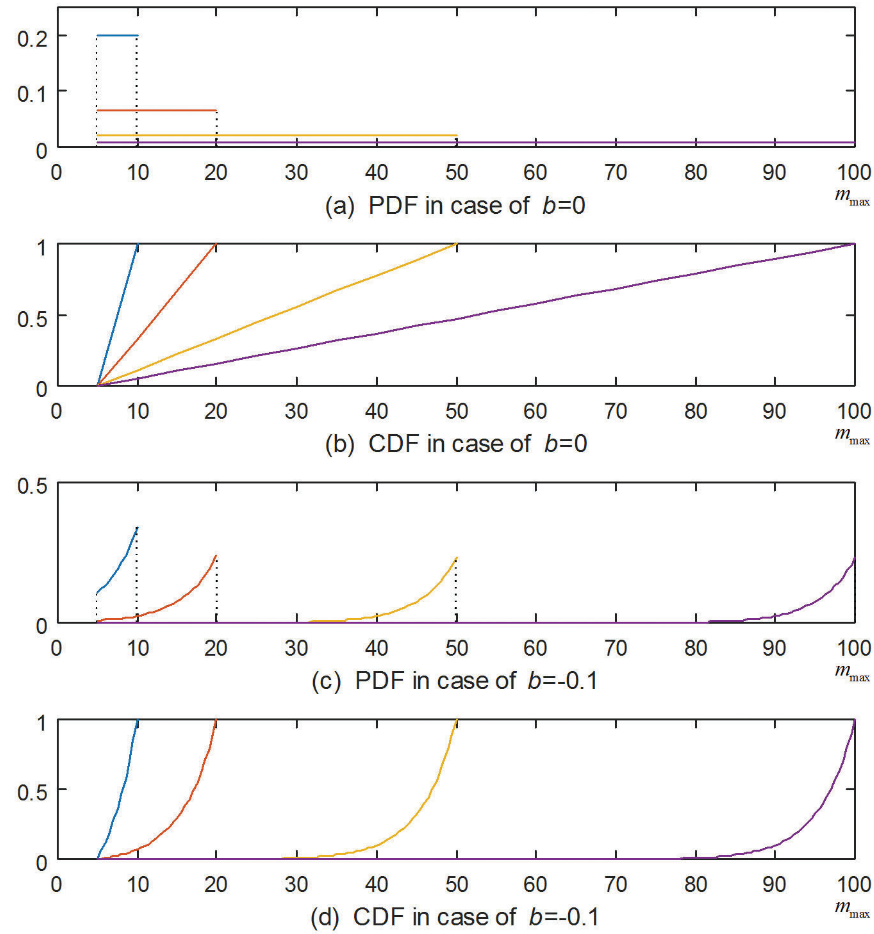

Figure 2. Some PDFs and CDFs for non-positive b-values

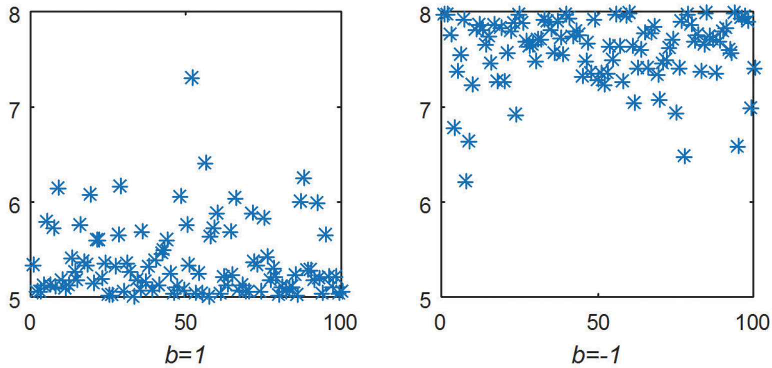

Figure 3. Distribution of events with different b-values

3. Kijko-Sellevoll functions

Let $M_1, M_2, \ldots, M_n \in [m_{\min}, m_{\max}]$ be a set of random variables from the catalogue. We assume that these random variables are independently and identically distributed (iid) with CDF of F_M given by (3). Moreover, let $m_1, m_2, \ldots, m_n$ to be a sample of magnitudes having a CDF

$$\bar{F}_{M_n}(m) = \begin{cases} 0, & \text{for } m < m_{\min}, \\ \left[F_M(m)\right]^n & \text{for } m_{\min} \le m < m_{\max}, \\ 1, & \text{for } m_{\max} \le m, \end{cases} \tag{4}$$

for all $n > 0$.

It is not necessary to assume that the magnitudes are ordered. Actually, we are using here the maximum function, $\max(M_1, M_2, \ldots, M_n)$. The formula (4) can be expressed as

$$\begin{aligned} P\left(\max_i (M_i) \le m\right) &= P(M_1 \le m \wedge M_2 \le m \wedge \ldots \wedge M_n \le m) \\ &= \prod_{i=1}^{n} P(M_i \le m) \\ &= \left[P(M_i \le m)\right]^n \\ &= \left[F_M(m)\right]^n \\ &= \bar{F}_{M_n}(m). \end{aligned}$$

Similar way for the minimum function, $\min(M_1, M_2, \ldots, M_n)$, (4) results as

$$\begin{aligned} P\left(\min_i(M_i) \le m\right) &= 1 - P\left(\min_i(M_i) > m\right) \\ &= 1 - P\left(M_1 > m \wedge M_2 > m \wedge \ldots \wedge M_N > m\right) \\ &= 1 - \prod_{i=1}^{N} P\left(M_i > m\right) \\ &= 1 - \left[P\left(M_i > m\right)\right]^n \\ &= 1 - \left[1 - P\left(M_i \le m\right)\right]^n \\ &= 1 - \left[1 - F_M(m)\right]^n \\ &= \underline{F}_{M_n}(m) \end{aligned}$$

with a CDF

$$\underline{F}_{M_n}(m) = \begin{cases} 0, & \text{for } m < m_{\min}, \\ 1 - \left[1 - F_M(m)\right]^n & \text{for } m_{\min} \le m < m_{\max}, \\ 1, & \text{for } m_{\max} \le m, \end{cases} \tag{5}$$

for all $n > 0$.

We have showed in our earlier work (Haarala and Orosco, 2016a, 2018), that the expected value of the maximum $\bar{M}_{(\eta)}$ in case of positive β is

$$\begin{aligned} \beta\left(m_{\max} - E\left(\bar{M}_{(\eta)}\right)\right) &= f_\eta^{KS-1}\left(\beta\left(m_{\max} - m_{\min}\right)\right), \\ \beta\left(E\left(\bar{M}_{(\eta)}\right) - m_{\min}\right) &= f_\eta^{KS-2}\left(\beta\left(m_{\max} - m_{\min}\right)\right), \end{aligned} \tag{6}$$

where f_η^{KS-1} is a Kijko-Sellevoll function 1 (KS-1)

$$f_\eta^{KS-1}(x) = \sum_{k=1}^{\infty} \frac{\left(1 - \exp[-x]\right)^k}{k + \eta} \tag{7}$$

and f_η^{KS-2} is a Kijko-Sellevoll function 2 (KS-2)

$$f_\eta^{KS-2}(x) = \eta \sum_{k=1}^{\infty} \frac{\left(1 - \exp[-x]\right)^k}{k\left(k + \eta\right)}. \tag{8}$$

These relationships are valid for all $\eta \in \mathbb{R}_+$ and for all $0 \le \beta\left(m_{\max} - m_{\min}\right) < \infty$ (Haarala and Orosco, 2018). The relation between KS-1 and KS-2 functions is

$$\beta(m_{\max}-m_{\min}) = f_\eta^{KS-1}\left(\beta(m_{\max}-m_{\min})\right) + f_\eta^{KS-2}\left(\beta(m_{\max}-m_{\min})\right). \tag{9}$$

Note that we have integer valued n in the CDFs (4) and (5), when we have a set of events. The real valued η is a useful feature in the applications, where the estimate of the number of events is a real value. For example, if we estimate 7.5 events by year, rounding this value into 7 or 8 we are producing a numerical bias for the results. It is to remember that the value 7.5 does not mean that there are really 7.5 events by year. The 7.5 is an average number of events by year, when we are considering a long interval of time. Our formulae make it possible to directly calculate those results without rounding.

We will give our proofs using variable η instead of n giving general results for the formulae. In the Appendix A it can be seen that η can be also negative even though the proofs are given only for positive real values, $\eta \in \mathbb{R}_+$.

4. The series for the expected values

Kijko-Sellevoll functions

First of all, we will show that the KS functions (7) and (8) are valid also on the interval $-\log(2) < \beta(\mathfrak{m}-m_{\min}) < 0$. Actually, our earlier proof (Haarala and Orosco, 2018) holds on this interval, if $\eta \in \mathbb{R}$. Because $\left(1-\exp\left[-\beta(\mathfrak{m}-m_{\min})\right]\right)^\eta$ is not defined generally when $\beta \in \mathbb{R}_-$ (it is defined only for $\eta \in \mathbb{N}$), we must consider $\left(\exp\left[-\beta(\mathfrak{m}-m_{\min})\right]-1\right)^\eta$ for all $\eta \in \mathbb{R}$. We have

$$\begin{aligned}
&\frac{\partial}{\partial\mathfrak{m}}\left[\left(\exp\left[-\beta(\mathfrak{m}-m_{\min})\right]-1\right)^\eta \frac{1}{\beta}\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta(\mathfrak{m}-m_{\min})\right]\right)^k}{k+\eta}\right] \\
&= \frac{\partial}{\partial\mathfrak{m}}\left[\frac{1}{\beta}\sum_{k=1}^{\infty}\frac{(-1)^k\left(\exp\left[-\beta(\mathfrak{m}-m_{\min})\right]-1\right)^{k+\eta}}{k+\eta}\right] \\
&= -\exp\left[-\beta(\mathfrak{m}-m_{\min})\right]\sum_{k=1}^{\infty}(-1)^k\left(\exp\left[-\beta(\mathfrak{m}-m_{\min})\right]-1\right)^{k+\eta-1} \\
&= \left(\exp\left[-\beta(\mathfrak{m}-m_{\min})\right]-1\right)^\eta \exp\left[-\beta(\mathfrak{m}-m_{\min})\right]\sum_{k=0}^{\infty}\left(1-\exp\left[-\beta(\mathfrak{m}-m_{\min})\right]\right)^k \\
&= \left(\exp\left[-\beta(\mathfrak{m}-m_{\min})\right]-1\right)^\eta.
\end{aligned} \tag{10}$$

The $\sum_{k=0}^{\infty}\left(1-\exp\left[-\beta(\mathfrak{m}-m_{\min})\right]\right)^k$ is a geometric series which gives $1/\exp\left[-\beta(\mathfrak{m}-m_{\min})\right]$ when $-\log(2) < \beta(\mathfrak{m}-m_{\min}) < 0$. (Actually, the convergence interval is $-\log(2) < \beta(\mathfrak{m}-m_{\min}) < \infty$, but we consider only the negative part since the proof of (10) is different when the positive part is considered.) This geometric series diverges at $\beta(\mathfrak{M}-m_{\min})=-\log(2)$. Thus, $\sum_{k=0}^{\infty}(-1)^k=1-1+1-1+-\cdots$. Owing to this it holds that $\sum_{k=0}^{\infty}\left(1-\exp\left[-\beta(\mathfrak{M}-m_{\min})\right]\right)^k=\exp\left[\beta(\mathfrak{M}-m_{\min})\right]$ in the interval $-\log(2)<\beta(\mathfrak{M}-m_{\min})<0$ and we have the limit $\exp\left[\beta(\mathfrak{M}-m_{\min})\right]\to 1/2$, when $\beta(\mathfrak{M}-m_{\min})\to-\log(2)$, we could define $\sum_{k=0}^{\infty}(-1)^k=1/2$. This limit could be seen like an expected value. Because of $\sum_{k=0}^{2n}(-1)^k=1$ and $\sum_{k=0}^{2n+1}(-1)^k=0$ for all $n=0,1,2,\ldots$; it is like the case of a coin, which has expected value $1/2$ when $n\to\infty$. This definition is related with the fact that the alternating series

$$\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta(\mathfrak{M}-m_{\min})\right]\right)^k}{k}=\sum_{k=1}^{\infty}\frac{(-1)^k}{k}=-\log(2)=\log\left(\frac{1}{2}\right)$$

converges when $\beta(\mathfrak{M}-m_{\min})=-\log(2)$.

The conclusion is that equality (10) holds for all $-\log(2)\le\beta(\mathfrak{M}-m_{\min})<0$ and it gives an integration formula

$$\begin{aligned}&\int\left(\exp\left[-\beta(\mathfrak{M}-m_{\min})\right]-1\right)^{\eta}d\mathfrak{M}\\&\quad=\left(\exp\left[-\beta(\mathfrak{M}-m_{\min})\right]-1\right)^{\eta}\frac{1}{\beta}\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta(\mathfrak{M}-m_{\min})\right]\right)^k}{k+\eta}+C.\end{aligned}\tag{11}$$

Applying this integration formula for the expected value, we have

$$\begin{aligned}E\left(\bar{M}_{(\eta)}\right)&=m_{\max}-\int_{m_{\min}}^{m_{\max}}\left(\frac{\exp\left[-\beta(\mathfrak{M}-m_{\min})\right]-1}{\exp\left[-\beta(m_{\max}-m_{\min})\right]-1}\right)^{\eta}d\mathfrak{M}\\&=m_{\max}-\frac{1}{\beta}\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta(m_{\max}-m_{\min})\right]\right)^k}{k+\eta}\quad\text{(KS-1)}\\&=m_{\max}-\frac{1}{\beta}\sum_{k=1}^{\infty}\left[\frac{1}{k}-\frac{\eta}{k(k+\eta)}\right]\left(1-\exp\left[-\beta(m_{\max}-m_{\min})\right]\right)^k\\&=m_{\min}+\frac{1}{\beta}\sum_{k=1}^{\infty}\frac{\eta\left(1-\exp\left[-\beta(m_{\max}-m_{\min})\right]\right)^k}{k(k+\eta)}\quad\text{(KS-2)}\end{aligned}\tag{12}$$

because $\sum_{k=1}^{\infty}z^k/k=-\log(1-z)$, $-1\le z<1$, and

$$\frac{1}{\beta}\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta(m_{\max}-m_{\min})\right]\right)^k}{k}=m_{\max}-m_{\min}$$

when $-\log(2)\le\beta(m_{\max}-m_{\min})<0$. The KS functions are alternating series in this interval.

The result (12) is the same than (6) with (8) in the non-negative interval. It means that we can use the relations

$$\beta\left[m_{\max}-E\left(\bar{M}\right)\right]=\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{k}}{k+\eta}=f_{\eta}^{KS-1}\left(\beta\left(m_{\max}-m_{\min}\right)\right)$$

$$\beta\left[E\left(\bar{M}\right)-m_{\min}\right]=\sum_{k=1}^{\infty}\frac{\eta\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{k}}{k\left(k+\eta\right)}=f_{\eta}^{KS-2}\left(\beta\left(m_{\max}-m_{\min}\right)\right)$$

for all $-\log(2)\le\beta\left(m_{\max}-m_{\min}\right)<\infty$ and $\eta\in\mathbb{R}_{+}$.

Extension for the first Kijko-Sellevoll function

It is much more complicated to solve the case when $\beta\left(m_{\max}-m_{\min}\right)<-\log(2)$. In this case, it is $\exp[-\beta(\mathfrak{M}-m_{\min})]-1>1$ (or in others words $-1<\left(1-\exp[-\beta(\mathfrak{M}-m_{\min})]\right)^{-1}<0$). In like manner as before, we get the geometric series as

$$\sum_{k=0}^{\infty}\left(\frac{1}{1-\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]}\right)^{k}=\frac{\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]-1}{\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]},$$

which is true for all $\mathfrak{M}\in\left]m_{\min}-\log(2)/\beta,\infty\right[$. Thus,

$$\begin{aligned}
&\int\left(\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]-1\right)^{\eta}d\mathfrak{M}\\
&=\int\left(\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]-1\right)^{\eta}\frac{\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]}{\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]-1}\sum_{k=0}^{\infty}\left(1-\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]\right)^{-k}d\mathfrak{M}\\
&=-\frac{1}{\beta}\int(-\beta)\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]\sum_{k=0}^{\infty}(-1)^{-k}\left(\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]-1\right)^{-k+\eta-1}d\mathfrak{M}\\
&=-\frac{1}{\beta}\sum_{k=0}^{\infty}\left\{(-1)^{k}\int(-\beta)\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]\left(\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]-1\right)^{-k+\eta-1}d\mathfrak{M}\right\}\\
&=-\frac{1}{\beta}\sum_{\substack{k=0\\k\neq\eta}}^{\infty}\frac{(-1)^{k}\left(\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]-1\right)^{-k+\eta}}{-k+\eta}+C.
\end{aligned}\tag{13}$$

There are two observations when $\eta\in\mathbb{N}$. Firstly, we have in the case $k=\eta-1$

$$\begin{aligned}
\int(-\beta)\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]d\mathfrak{M}&=\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]+C\\
&=\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]-1,
\end{aligned}$$

where we have set $C=-1$. Secondly, the integration in the case $k=\eta$ gives

$$\frac{(-1)^{k+1}}{\beta}\int\frac{(-\beta)\exp\left[-\beta(\mathfrak{M}-m_{\min})\right]}{\exp\left[-\beta(\mathfrak{M}-m_{\min})\right]-1}d\mathfrak{M}=\frac{(-1)^{k+1}}{\beta}\log\left(\exp\left[-\beta(\mathfrak{M}-m_{\min})\right]-1\right)+C. \tag{14}$$

Hence,

$$\int\left(\exp\left[-\beta(\mathfrak{M}-m_{\min})\right]-1\right)^{\eta}d\mathfrak{M}=$$
$$\frac{1}{\beta}\sum_{k=0}^{\infty}\left\langle\frac{(-1)^{k}\left(\exp\left[-\beta(m_{\max}-m_{\min})\right]-1\right)^{-k+\eta}}{k-\eta}\middle|(-1)^{\eta+1}\log\left(\exp\left[-\beta(m_{\max}-m_{\min})\right]-1\right)\right\rangle_{(k=\eta)}+C, \tag{15}$$

where $\langle x|y\rangle_{(c)}$ is a switch function giving x, if c is false, and y, if c is true.

When we worked with KS functions in negative side, we had $0<m_{\max}-m_{\min}\leq-\log(2)/\beta$ (or in other words $m_{\min}<m_{\max}\leq m_{\min}-\log(2)/\beta$). This means that the $m_{\max}$ must be close enough to $m_{\min}$ when we integrate over the interval $[m_{\min},m_{\max}]$. If the difference between $m_{\max}-m_{\min}$ is bigger, we have $m_{\min}<m_{\min}-\log(2)/\beta<m_{\max}<\infty\Leftrightarrow-\infty<\beta(m_{\max}-m_{\min})<-\log(2)$. Integrating over the interval $\left[m_{\min}-\log(2)/\beta,m_{\max}\right]$, we get

$$\int_{m_{\min}-\log(2)/\beta}^{m_{\max}}\left(\frac{\exp\left[-\beta(\mathfrak{M}-m_{\min})\right]-1}{\exp\left[-\beta(m_{\max}-m_{\min})\right]-1}\right)^{\eta}d\mathfrak{M}$$
$$=\frac{1}{\beta}\left(\exp\left[-\beta(m_{\max}-m_{\min})\right]-1\right)^{-\eta}\sum_{\substack{k=0\\k\neq\eta}}^{\infty}\frac{(-1)^{k}\left[\left(\exp\left[-\beta(m_{\max}-m_{\min})\right]-1\right)^{-k+\eta}-1\right]}{k-\eta}. \tag{16}$$

Due to $\left(y^{h}-1\right)/h\to\log(y)$, when $h\to0$, we can get the limit

$$\frac{(-1)^{k+1}}{\beta}\frac{\left(\exp\left[-\beta(m_{\max}-m_{\min})\right]-1\right)^{-k+\eta}-1}{-k+\eta}\xrightarrow{\eta\to k}\frac{(-1)^{k+1}}{\beta}\log\left(\exp\left[-\beta(m_{\max}-m_{\min})\right]-1\right). \tag{17}$$

This is the same as (14). Hence, we say that the series in (16) holds for all $\eta\in\mathbb{R}_{+}$, where we replace the discontinuity term by (14) in the case $k=\eta$ of our calculus.

Using integration formula (11), the integration over $\left]m_{\min},m_{\min}-\log(2)/\beta\right]$ gives

$$\int_{m_{\min}}^{m_{\min}-\log(2)/\beta}\left(\frac{\exp\left[-\beta(\mathfrak{M}-m_{\min})\right]-1}{\exp\left[-\beta(m_{\max}-m_{\min})\right]-1}\right)^{\eta}d\mathfrak{M}=\frac{1}{\beta}\left(\exp\left[-\beta(m_{\max}-m_{\min})\right]-1\right)^{-\eta}\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k+\eta}.$$

The final result of the integral can be written as

$$E\left(\bar{M}_{(\eta)}\right) = m_{\max} - \int_{m_{\min}}^{m_{\max}} \left(\frac{\exp\left[-\beta\left(\mathfrak{M} - m_{\min}\right)\right] - 1}{\exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right] - 1} \right)^{\eta} d\mathfrak{M}$$

$$= m_{\max} - \frac{1}{\beta}\left(\exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right] - 1\right)^{-\eta} \left\{ \sum_{k=1}^{\infty} \frac{(-1)^k}{k+\eta} + \sum_{k=0}^{\infty} \left\langle \frac{(-1)^k \left[\left(\exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right] - 1\right)^{-k+\eta} - 1\right]}{k-\eta} \middle| (-1)^{\eta+1} \log\left(\exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right] - 1\right) \right\rangle_{(k=\eta)} \right\}.$$

So, we call the function

$$f_{\eta}^{EKS-1}\left(\beta\left(m_{\max} - m_{\min}\right)\right) = \left(\exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right] - 1\right)^{-\eta} \left\{ \sum_{k=1}^{\infty} \frac{(-1)^k}{k+\eta} + \sum_{k=0}^{\infty} \left\langle \frac{(-1)^k \left[\left(\exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right] - 1\right)^{-k+\eta} - 1\right]}{k-\eta} \middle| (-1)^{\eta+1} \log\left(\exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right] - 1\right) \right\rangle_{(k=\eta)} \right\}. \quad (18)$$

as an Extension for the Kijko-Sellevoll function 1 (EKS-1). This function is valid, when $-\infty < \beta\left(m_{\max} - m_{\min}\right) < -\log(2)$.

This function (18) does not look like a KS-1 function, but it is a reflection of it (Appendix A). Also, we could show that EKS-1 function yields

$$f_{n}^{EKS-1}\left(\beta\left(m_{\max} - m_{\min}\right)\right) = \frac{\beta\left(m_{\max} - m_{\min}\right) - \sum_{k=1}^{n} \frac{\left(1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]\right)^k}{k}}{\left(1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]\right)^n} \quad (19)$$

when $n \in \mathbb{N}$. This expression was found anterior work (Haarala and Orosco, 2016a) by showing

$$f_{n}^{KS-1}\left(\beta\left(m_{\max} - m_{\min}\right)\right) = \sum_{k=1}^{\infty} \frac{\left(1 - \exp[-x]\right)^k}{k+\eta} = \frac{\beta\left(m_{\max} - m_{\min}\right) - \sum_{k=1}^{n} \frac{\left(1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]\right)^k}{k}}{\left(1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]\right)^n}.$$

The proof of relation (19) is given in Appendix A.

Even though we have the discontinuity term in the series, there is no discontinuity as we showed above. In the numerical calculus, it is to replace the discontinuous term with the logarithmic term (14). Because we use the acceleration method to calculate the series, we do not need to mind this correction if η is bigger than the number of terms in the accelerated sum (for example, $\eta > 20$ in double precision systems).

There is an alternative way to solve the problem of the expected value in the case of the negative b -value; we will show that in the Appendix B.

Extension for the second Kijko-Sellevoll function

The Extension for the Kijko-Sellevoll function 2 (EKS-2) could be found by

$$f_{\eta}^{EKS-2}\left(\beta\left(m_{\max}-m_{\min}\right)\right)=\beta\left(m_{\max}-m_{\min}\right)-f_{\eta}^{EKS-1}\left(\beta\left(m_{\max}-m_{\min}\right)\right).$$

To find the EKS-2, the $-\beta\left(m_{\max}-m_{\min}\right)$ and $\log(2)$ have the series

$$\begin{aligned}-\beta\left(m_{\max}-m_{\min}\right)&=\log\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)+\log\left(1-\frac{1}{1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]}\right)\\&=\log\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)-\sum_{k=1}^{\infty}\frac{(-1)^{-k}\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{-k}}{k}\end{aligned}\tag{20}$$

and

$$\log(2)=\log\left(1-(-1)\right)=-\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k},\tag{21}$$

respectively. Using (21), we get from (20)

$$\begin{aligned}n_{\max}-m_{\min}&=-\frac{1}{\beta}\log\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)\\&\quad+\frac{1}{\beta}\frac{\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k}+\left\{\sum_{k=1}^{\infty}\frac{(-1)^{k}\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{-k+\eta}}{k}-\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k}\right\}}{\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{\eta}}\\&=-\frac{1}{\beta}\log\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)\\&\quad+\frac{1}{\beta}\frac{\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k}+\sum_{k=1}^{\infty}\frac{(-1)^{k}\left[\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{-k+\eta}-1\right]}{k}}{\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{\eta}}.\end{aligned}$$

Finally, the expected value yields to

$$E\left(\bar{M}_{(\eta)}\right)=m_{\min}+\left(m_{\max}-m_{\min}\right)-\frac{1}{\beta}\frac{\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k+\eta}+\sum_{\substack{k=0\\k\neq\eta}}^{\infty}\frac{(-1)^{k}\left[\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{-k+\eta}-1\right]}{k-\eta}}{\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{\eta}}$$

$$=m_{\min}+\frac{1}{\beta}\left\{-\log\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)+\frac{1-\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{-\eta}}{\eta}\right.$$

$$+\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{-\eta}\sum_{k=1}^{\infty}\frac{\eta(-1)^{k}}{k(k+\eta)}$$

$$\left.-\sum_{\substack{k=1\\k\neq\eta}}^{\infty}\frac{\eta(-1)^{k}\left[\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{-k}-\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{-\eta}\right]}{k(k-\eta)}\right\}$$

where we need to replace the discontinuity term of the series by

$$\lim_{k\to\eta}\left\{\frac{\eta(-1)^{k}\left[\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{-k}-\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{-\eta}\right]}{k(k-\eta)}\right\}$$

$$=-\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{-\eta}\lim_{k\to\eta}\left\{\frac{\eta(-1)^{k}\left[\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{-k+\eta}-1\right]}{k(-k+\eta)}\right\}$$

$$=(-1)^{\eta+1}\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{-\eta}\log\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right).$$

when $n\in\mathbb{N}$. The Extension for the Kijko-Sellevoll function 2 (EKS-2) is now

$$f_{\eta}^{EKS-2}\left(\beta\left(m_{\max}-m_{\min}\right)\right)=-\log\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)+\frac{1-\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{-\eta}}{\eta}$$

$$+\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{-\eta}\left\{\sum_{k=1}^{\infty}\frac{\eta(-1)^{k}}{k(k+\eta)}\right.$$

$$\left.-\sum_{k=1}^{\infty}\left\langle\frac{\eta(-1)^{k}\left[\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{-k+\eta}-1\right]}{k(k-\eta)}\middle|(-1)^{\eta+1}\log\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)\right\rangle_{(k=\eta)}\right\}$$

for $\eta\in\mathbb{R}_{+}$.

Uniform distribution

The Uniform Distribution function results are well known, we will show here how we can also get them from the GGR CDF. Let start with the KS-2 function

$$E\left(\bar{M}_{(\eta)}\right)-m_{\min}=\frac{\eta}{\beta}\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{k}}{k\left(k+\eta\right)}. \tag{22}$$

Since

$$1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]=\beta\left(m_{\max}-m_{\min}\right)\left\{1+\sum_{j=2}^{\infty}\frac{\left[-\beta\left(m_{\max}-m_{\min}\right)\right]^{j-1}}{j!}\right\} \tag{23}$$

the equation (22) gives

$$E\left(\bar{M}_{(\eta)}\right)-m_{\min}=\frac{\eta}{\eta+1}\left(m_{\max}-m_{\min}\right)\left\{1+\sum_{j=2}^{\infty}\frac{\left[-\beta\left(m_{\max}-m_{\min}\right)\right]^{j-1}}{j!}\right\}$$
$$+\eta\sum_{k=2}^{\infty}\frac{\beta^{k-1}\left(\left(m_{\max}-m_{\min}\right)\left\{1+\sum_{j=2}^{\infty}\frac{\left[-\beta\left(m_{\max}-m_{\min}\right)\right]^{j-1}}{j!}\right\}\right)^{k}}{k\left(k+\eta\right)}.$$

If $\beta\to 0$, then we have the KS-2 estimator for the Expected value as

$$E\left(\bar{M}_{(\eta)}\right)=m_{\min}+\frac{\eta}{\eta+1}\left(m_{\max}-m_{\min}\right) \tag{24}$$

which is known as an unbiased estimator for the maximum of the Uniform Distribution function in the form

$$m_{\max}=\frac{\eta+1}{\eta}\left(E\left(\bar{M}_{(\eta)}\right)-m_{\min}\right)+m_{\min}.$$

The KS-1 estimator at $\beta=0$ can be directly got by

$$E\left(\bar{M}_{(\eta)}\right)=m_{\min}+\left(m_{\max}-m_{\min}\right)-\left[\left(m_{\max}-m_{\min}\right)-\frac{\eta}{\eta+1}\left(m_{\max}-m_{\min}\right)\right]$$
$$=m_{\max}-\frac{1}{\eta+1}\left(m_{\max}-m_{\min}\right).$$

Now we have shown all possible cases to calculate the expected value. We give new definitions to the KS functions:

$$f_\eta^{KS_{\max}}(x)=\begin{cases} f_\eta^{EKS-1}(x), & \text{when } -\infty\le x<-\log(2),\\ f_\eta^{KS-1}(x), & \text{when } -\log(2)\le x\le\infty,\end{cases}$$

$$f_\eta^{KS_{\min}}(x)=\begin{cases} f_\eta^{EKS-2}(x), & \text{when } -\infty\le x<-\log(2),\\ f_\eta^{KS-2}(x), & \text{when } -\log(2)\le x\le\infty\end{cases}$$

and $\eta\in\mathbb{R}_+$. The name KS_{max} associates better the KS function or its extension to the maximum, because KS-1 and EKS-1 are measures of the distance from the maximum to the expected value. Similarly, because KS-2 and EKS-2 are measures of the distance from the minimum to the expected value, the name KS_{min} associates the KS function or its extension to the minimum. We can see the examples of the $f_5^{KS_{\max}}$ and $f_5^{KS_{\min}}$ in Figure 4.

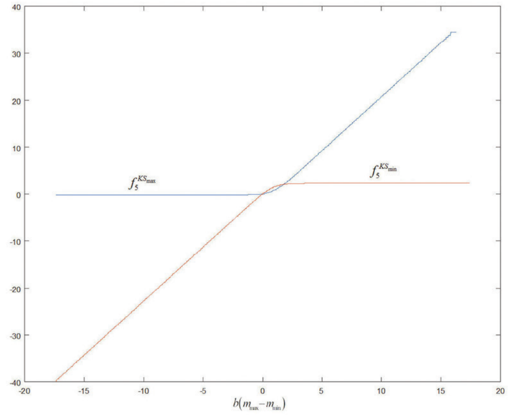

Figure 4. Example of the KS functions

5. Series for the variance

The third Kijko-Sellevoll function

As we saw in the case of the expected values above, it is only a technical detail to prove that the KS functions work also in the negative side. If we assume that $\eta\in\mathbb{N}$, we need no changes to the earlier proofs. We can see that in this case the KS-3 is valid in the interval $-\log(2)\le\beta(m_{\max}-m_{\min})\le\infty$, but we need to assume $\eta\in\mathbb{R}_+$ in more general case.

Let's start with

$$\begin{aligned}
&\frac{\partial}{\partial \mathfrak{m}}\left[\left(\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]-1\right)^{\eta}\frac{1}{\beta}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{(-1)^{k+j}\left(\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]-1\right)^{k+j}}{(\eta+k)(\eta+k+j)}\right]\\
&=\frac{\partial}{\partial \mathfrak{m}}\left[-\frac{1}{\beta}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{(-1)^{k+j-1}\left(\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]-1\right)^{\eta+k+j}}{(\eta+k)(\eta+k+j)}\right]\\
&=\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{(-1)^{k+j-1}\left(\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]-1\right)^{\eta+k+j-1}}{(\eta+k)} \qquad (25)\\
&=\sum_{k=1}^{\infty}\frac{(-1)^{k}\left(\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]-1\right)^{\eta+k}\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\sum_{j=0}^{\infty}\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{j}}{\eta+k}\\
&=\left(\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]-1\right)^{\eta}\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{k}}{\eta+k},
\end{aligned}$$

where the geometric series gives (when $\mathfrak{m}\in\left[m_{\min},m_{\min}-\log(2)/\beta\right[$)

$$\sum_{j=0}^{\infty}\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{j}=\frac{1}{\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]}.$$

Thus,

$$\begin{aligned}
&\int\left(\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]-1\right)^{\eta}\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{k}}{\eta+k}\,d\mathfrak{m}\\
&\qquad=\left(\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]-1\right)^{\eta}\frac{1}{\beta}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{k+j}}{(\eta+k)(\eta+k+j)}+C,
\end{aligned}$$

when $\eta\in\mathbb{R}_{+}$ and $-\log(2)\le\beta\left(m_{\max}-m_{\min}\right)<0$. Following our earlier work (Haarala and Orosco, 2016), the second moment can be integrated by parts as

$$\begin{aligned}
E\left(\bar{M}_{(\eta)}^{2}\right) &= \int_{m_{\min}}^{m_{\max}} \mathfrak{m}^{2}\, dF_{M_n}\left(\mathfrak{m}\right) \\
&= m_{\max}^{2} - 2\int_{m_{\min}}^{m_{\max}} \mathfrak{m}\, F_{M_n}\left(\mathfrak{m}\right)\, d\mathfrak{m} \\
&= m_{\max}^{2} - 2m_{\max}\int_{m_{\min}}^{m_{\max}} F_{M_n}\left(\mathfrak{m}\right) d\mathfrak{m} + 2\int_{m_{\min}}^{m_{\max}}\int_{m_{\min}}^{\mathfrak{m}} F_{M_n}\left(y\right) dy\, d\mathfrak{m} \\
&= \left[m_{\max} - \int_{m_{\min}}^{m_{\max}} F_{M_n}\left(\mathfrak{m}\right) d\mathfrak{m}\right]^{2} + 2\int_{m_{\min}}^{m_{\max}}\int_{m_{\min}}^{\mathfrak{m}} F_{M_n}\left(y\right) dy\, d\mathfrak{m} - \left[\int_{m_{\min}}^{m_{\max}} F_{M_n}\left(\mathfrak{m}\right) d\mathfrak{m}\right]^{2} \\
&= \left[E\left(\bar{M}_{(\eta)}\right)\right]^{2} + 2\int_{m_{\min}}^{m_{\max}}\int_{m_{\min}}^{\mathfrak{m}} \left(\frac{1-\exp\left[-\beta\left(y - m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]}\right)^{\eta} dy\, d\mathfrak{m} \\
&\quad - \left[\int_{m_{\min}}^{m_{\max}} \left(\frac{1-\exp\left[-\beta\left(\mathfrak{m} - m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]}\right)^{\eta} d\mathfrak{m}\right]^{2}.
\end{aligned} \tag{26}$$

Hence,

$$\begin{aligned}
Var\left(\bar{M}_{(\eta)}\right) &= E\left(\bar{M}_{(\eta)}^{2}\right) - \left[E\left(\bar{M}_{(\eta)}\right)\right]^{2} \\
&= \frac{2}{\beta}\frac{\displaystyle\int_{m_{\min}}^{m_{\max}} \left[\exp\left[-\beta\left(\mathfrak{m} - m_{\min}\right)\right]-1\right]^{\eta} \sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(\mathfrak{m} - m_{\min}\right)\right]\right)^{k}}{\eta + k} d\mathfrak{m}}{\left[\exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]-1\right]^{\eta}} \\
&\quad - \left[\frac{1}{\beta}\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]\right)^{k}}{\eta + k}\right]^{2} \\
&= \frac{2}{\beta^{2}}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]\right)^{k+j}}{\left(\eta + k\right)\left(\eta + k + j\right)} - \frac{1}{\beta^{2}}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]\right)^{k+j}}{\left(\eta + k\right)\left(\eta + j\right)},
\end{aligned} \tag{27}$$

because

$$\begin{aligned}
\left[\int_{m_{\min}}^{m_{\max}} \left(\frac{1-\exp\left[-\beta\left(\mathfrak{m} - m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]}\right)^{\eta} d\mathfrak{m}\right]^{2} &= \left[\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(\mathfrak{m} - m_{\min}\right)\right]\right)^{k}}{\eta + k}\right]\left[\sum_{j=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(\mathfrak{m} - m_{\min}\right)\right]\right)^{j}}{\eta + j}\right] \\
&= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(\mathfrak{m} - m_{\min}\right)\right]\right)^{k+j}}{\left(\eta + k\right)\left(\eta + j\right)}.
\end{aligned}$$

We must point out that we integrate $\left[(\exp(\cdot)-1)/(\exp(\cdot)-1)\right]^{\eta}$ instead of $\left[(1-\exp(\cdot))/(1-\exp(\cdot))\right]^{\eta}$ because the exponent function $[\cdot]^{\eta}$ does not exist in the real axes for all $\eta \in \mathbb{R}_{+}$.

The obtained result (27) is the same as that obtained for positive b (Haarala and Orosco, 2016b), so the KS-3 holds in the interval $-\log(2) \le \beta(m_{\max}-m_{\min}) < \infty$ just like other KS functions. Because of the series in (27) have an absolute convergence in the open interval $-\log(2) < \beta(m_{\max}-m_{\min}) < \infty$, so we can rearrange those series. Thus, we can calculate the variance as

$$Var\left(\bar{M}_{(\eta)}\right) = \frac{1}{\beta^2}\sum_{k=2}^{\infty}\left\{\sum_{j=1}^{k-1}\frac{1}{\eta+j}\right\}\frac{2\eta}{2\eta+k}\frac{\left(1-\exp\left[-\beta(m_{\max}-m_{\min})\right]\right)^k}{\eta+k} = \frac{1}{\beta^2} f_{\eta}^{KS-3}\left(\beta(m_{\max}-m_{\min})\right). \quad (28)$$

We need to check yet the point at $-\log(2)$. Because we know that (Haarala and Orosco, 2016b)

$$Var\left(\bar{M}_{(n)}\right) = \frac{1}{\beta^2}\sum_{k=2}^{\infty}\left\{\sum_{j=1}^{k-1}\frac{1}{n+j}\right\}\frac{2n}{2n+k}\frac{\left(1-\exp\left[-\beta(m_{\max}-m_{\min})\right]\right)^k}{n+k} \overset{\beta(m_{\max}-m_{\min})\to\infty}{\longrightarrow} \frac{H_n^{(2)}}{\beta^2} \le \frac{\pi^2}{6\beta^2}$$

for all $n \in \mathbb{N}$, where $H_n^{(2)} = \sum_{k=1}^{n} k^{-2}$ is a Harmonic Number of order 2. This shows that the (28) has an absolute convergence at $-\log(2)$, so it converges at the same point. So, the variance (and the KS-3) holds for all $-\log(2) \le \beta(m_{\max}-m_{\min}) < \infty$.

It is worth noting that a General Harmonic Number of order 2 can be defined as

$$H_{\eta}^{(2)} = \sum_{k=2}^{\infty}\left\{\sum_{j=1}^{k-1}\frac{1}{\eta+j}\right\}\frac{2\eta}{(2\eta+k)(\eta+k)} = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{\eta+j-k}{(\eta+k)(\eta+j)(\eta+j+k)}.$$

It holds $H_{\eta}^{(2)} = H_n^{(2)}$ for all $\eta = n = 0, 1, 2, \ldots$.

Extension for the third Kijko-Sellevoll function

To find extension for the KS-3 in the case $\beta(\mathfrak{m}-m_{\min}) < -\log(2)$, we get

$$Var\left(M_{(\eta)}\right) = -\frac{2}{\beta}\frac{\displaystyle\int_{m_{\min}-\log(2)/\beta}^{m_{\max}}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\exp\left[-\beta(\mathfrak{m}-m_{\min})\right]-1\right)^{\eta-k}}{\eta-k}d\mathfrak{m}}{\left[\exp\left[-\beta(m_{\max}-m_{\min})\right]-1\right]^{\eta}}$$
$$+\frac{2}{\beta}\frac{\displaystyle\int_{m_{\min}}^{m_{\min}-\log(2)/\beta}\sum_{k=1}^{\infty}\frac{(-1)^k\left(\exp\left[-\beta(\mathfrak{m}-m_{\min})\right]-1\right)^{\eta+k}}{\eta+k}d\mathfrak{m}}{\left[\exp\left[-\beta(m_{\max}-m_{\min})\right]-1\right]^{\eta}} - \left[\frac{1}{\beta}f_{\eta}^{EKS-1}\left(\beta(m_{\max}-m_{\min})\right)\right]^2.$$

We applied the same procedure which resulted in (27) except that here the integral has divided into two parts. The second integral can be got directly (alike the function KS-3)

$$\frac{2}{\beta}\frac{\displaystyle\int_{m_{\min}}^{m_{\min}-\log(2)/\beta}\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{\eta+k}}{\eta+k}d\mathfrak{m}}{\left[\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right]^{\eta}}=\frac{2}{\beta^{2}}\frac{\displaystyle\sum_{k=2}^{\infty}(-1)^{k}\left\{\sum_{j=1}^{k-1}\frac{1}{\eta+j}\right\}\frac{1}{\eta+k}}{\left[\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right]^{\eta}}.$$

With the first integral, we start with the geometric series

$$\sum_{j=0}^{\infty}\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{-j}=\frac{\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]-1}{\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]}. \tag{29}$$

Integrating all terms which has $k\neq\eta$, we find

$$\begin{aligned}
&-\int\sum_{\substack{k=0\\k\neq\eta}}^{\infty}\frac{(-1)^{k}\left(\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]-1\right)^{\eta-k}}{\eta-k}d\mathfrak{m}\\
&=-\int\left\{\sum_{\substack{k=0\\k\neq\eta}}^{\infty}\frac{(-1)^{-k}\left(\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]-1\right)^{\eta-k}}{\eta-k}\frac{\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]}{\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]-1}\sum_{j=0}^{\infty}\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{-j}\right\}d\mathfrak{m}\\
&=\frac{1}{\beta}\int(-\beta)\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\sum_{\substack{k=0\\k\neq\eta}}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{-k-j}\left(\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]-1\right)^{\eta-k-j-1}}{\eta-k}d\mathfrak{m}\\
&=\frac{1}{\beta}\sum_{\substack{k=0\\k\neq\eta}}^{\infty}\sum_{\substack{j=0\\j\neq\eta-k}}^{\infty}\frac{(-1)^{k+j}\left(\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]-1\right)^{\eta-k-j}}{(\eta-k)(\eta-k-j)}+C,
\end{aligned}$$

where the term $j=\eta-k$ must be calculated as

$$\frac{1}{\beta}\int\frac{(-1)^{\eta}}{(\eta-k)}\frac{(-\beta)\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]}{\left(\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]-1\right)}d\mathfrak{m}=\frac{1}{\beta}\frac{(-1)^{\eta}\log\left(\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]-1\right)}{\eta-k}+C.$$

Thus,

$$\begin{aligned}
&\int\frac{(-1)^{k}\left(\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]-1\right)^{-k+\eta}}{k-\eta}d\mathfrak{m}=\\
&\frac{1}{\beta}\sum_{\substack{k=0\\k\neq\eta}}^{\infty}\sum_{j=0}^{\infty}\left\langle\frac{(-1)^{k+j}\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{\eta-k-j}}{(\eta-k)(\eta-k-j)}\middle|\frac{(-1)^{\eta}\log\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)}{\eta-k}\right\rangle_{(j=\eta-k)}+C.
\end{aligned}$$

Similar way as before, the variance yields

$$Var\left(M_{(\eta)}\right)=\frac{1}{\beta^2}\left\{\frac{2\sum_{\substack{k=0\\k\neq\eta}}^{\infty}\sum_{j=0}^{\infty}\left\langle\frac{(-1)^{k+j}\left(\exp\left[-\beta\left(m_{max}-m_{min}\right)\right]-1\right)^{\eta-k-j}}{(\eta-k)(\eta-k-j)}\middle|\frac{(-1)^{\eta}\log\left(\exp\left[-\beta\left(m_{max}-m_{min}\right)\right]-1\right)}{\eta-k}\right\rangle_{(j=\eta-k)}+2\sum_{k=2}^{\infty}(-1)^k\left\{\sum_{j=1}^{k-1}\frac{1}{\eta+j}\right\}\frac{1}{\eta+k}}{\left[\exp\left[-\beta\left(m_{max}-m_{min}\right)\right]-1\right]^{\eta}}\right\}-\left[f_{\eta}^{EKS-1}\left(\beta\left(m_{max}-m_{min}\right)\right)\right]^2$$
$$=\frac{1}{\beta^2}f_{\eta}^{EKS-3}\left(\beta\left(m_{max}-m_{min}\right)\right)$$

We call the function f_{η}^{EKS-3} as an Extension for the Kijko-Sellevoll function 3 (EKS-3).

As it can be seen, we did not solve the problem at the discontinuity point $k=\eta$ (or $j\neq\eta-k$ in the case of EKS-3) as we made before. Even the formula of EKS-3 is valid in any neighborhood of k, it will be unstable to calculate numerically when $\eta\approx k$.

Variance for the Uniform distribution

We have proved above, that KS-3 is valid when $-\log(2)\leq\beta\left(m_{max}-m_{min}\right)<\infty$, so they are valid in the neighborhood of zero. We can apply the (23) in to the(28), so

$$\frac{1}{\beta^2}\sum_{k=2}^{\infty}\frac{2\eta}{2\eta+k}\left\{\sum_{j=1}^{k-1}\frac{1}{\eta+j}\right\}\frac{\left(\beta\left(m_{max}-m_{min}\right)\left\{1+\sum_{j=2}^{\infty}\frac{\left[-\beta\left(m_{max}-m_{min}\right)\right]^{j-1}}{j!}\right\}\right)^k}{\eta+k}$$
$$=\frac{\eta}{\eta+1}\frac{1}{\eta+1}\frac{\left(m_{max}-m_{min}\right)^2\left\{1+\sum_{j=2}^{\infty}\frac{\left[-\beta\left(m_{max}-m_{min}\right)\right]^{j-1}}{j!}\right\}^2}{\eta+2}$$
$$+\sum_{k=3}^{\infty}\frac{2\eta}{2\eta+k}\left\{\sum_{j=1}^{k-1}\frac{1}{\eta+j}\right\}\frac{\beta^{k-2}\left(\left(m_{max}-m_{min}\right)\left\{1+\sum_{j=2}^{\infty}\frac{\left[-\beta\left(m_{max}-m_{min}\right)\right]^{j-1}}{j!}\right\}\right)^k}{\eta+k}$$
$$\rightarrow\frac{\eta}{\eta+2}\left(\frac{m_{max}-m_{min}}{\eta+1}\right)^2.$$

This gives $\left(m_{max}-m_{min}\right)^2/12$ in the case $\eta=1$.

6. Some analysis of the GR distributed data

Let assume that $m_{min}=m_{max}$. Then the expected value gives

$$E\left(M_{(\eta)}\right)=m_{max}-\int_{m_{min}}^{m_{max}}\left[F_M\left(\mathfrak{m}\right)\right]^{\eta}d\mathfrak{m}=m_{max}=m_{min}.$$

This trivial result shows that the expected value is constant for all $\eta\in\mathbb{R}_+$.

Symmetrically distributed data (Uniform distribution case)

Let $\eta_1, \eta_2 \in \mathbb{R}_+$ to be fixed. We have now

$$m_{\min} = (\eta_1 + 1) E\left(\bar{M}_{(\eta_1)}\right) - \eta_1 m_{\max} ,$$
$$m_{\min} = (\eta_2 + 1) E\left(\bar{M}_{(\eta_2)}\right) - \eta_2 m_{\max} ,$$

thus,

$$\begin{aligned} m_{\max} &= \frac{\eta_1 + 1}{\eta_1 - \eta_2} E\left(\bar{M}_{(\eta_1)}\right) - \frac{\eta_2 + 1}{\eta_1 - \eta_2} E\left(\bar{M}_{(\eta_2)}\right), \\ m_{\min} &= -\eta_2 \frac{\eta_1 + 1}{\eta_1 - \eta_2} E\left(\bar{M}_{(\eta_1)}\right) + \eta_1 \frac{\eta_2 + 1}{\eta_1 - \eta_2} E\left(\bar{M}_{(\eta_2)}\right). \end{aligned} \tag{30}$$

Because $m_{\min} \le m_{\max}$, it indicates

$$-\eta_2 \frac{\eta_1 + 1}{\eta_1 - \eta_2} E\left(\bar{M}_{(\eta_1)}\right) + \eta_1 \frac{\eta_2 + 1}{\eta_1 - \eta_2} E\left(\bar{M}_{(\eta_2)}\right) \le \frac{\eta_1 + 1}{\eta_1 - \eta_2} E\left(\bar{M}_{(\eta_1)}\right) - \frac{\eta_2 + 1}{\eta_1 - \eta_2} E\left(\bar{M}_{(\eta_2)}\right)$$
$$\Updownarrow$$
$$E\left(\bar{M}_{(\eta_2)}\right) \le E\left(\bar{M}_{(\eta_1)}\right)$$

Thus, $\eta_2 \le \eta_1$ because of the expected value function is increasing.

The formulae (30) shows that if we can find estimators for the expected values $E\left(\bar{M}_{(\eta_1)}\right)$ and $E\left(\bar{M}_{(\eta_2)}\right)$, we can calculate the estimates for the $m_{\max}$ and $m_{\min}$ quite simple way. We will give an example in the section 8 how to use the formulae (30).

It is not to be forgotten that all expected values lie between then minimum and maximum as

$$m_{\min} \le E\left(\bar{M}_{(\eta)}\right) = m_{\min} + \frac{\eta}{\eta + 1}(m_{\max} - m_{\min}) \le m_{\max} \tag{31}$$

for all $\eta \in \mathbb{R}_+$. The limits are $E\left(\bar{M}_{(0)}\right) = \lim_{\eta \to 0+} E\left(\bar{M}_{(\eta)}\right) = m_{\min}$ and $E\left(\bar{M}_{(\infty)}\right) = \lim_{\eta \to \infty} E\left(\bar{M}_{(\eta)}\right) = m_{\max}$. If $E\left(\bar{M}_{(\eta)}\right) = \infty$ or $E\left(\bar{M}_{(\eta)}\right) = -\infty$ for all $\eta \in \mathbb{R}_+$ or $E\left(\bar{M}_{(\eta_2)}\right) = -\infty$ and $E\left(\bar{M}_{(\eta_1)}\right) = \infty$ of some $\eta_2 < \eta_1$, then the limits are unbounded.

If both expected values $E\left(\bar{M}_{(\eta_1)}\right)$ and $E\left(\bar{M}_{(\eta_2)}\right)$ are bounded, the right-hand side in both equations are bounded and the data has bounded limits. We can see also from (30) that if one expected value is bounded and another is unbounded, the limits are unbounded. But if it happens, there exists only one bounded expected value because other way we can find two bounded expected values showing bounded limits.

We have seen that two bounded expected values at distinct points guaranties bounded limits for the data in case of $\beta = 0$.

We will see next that the maximum estimates of $m_{\max}$ and $m_{\min}$ are bounded in case of the uniform distribution. Suppose that we have n events $m_1, m_2, \ldots, m_n$. Without losing generality, m_n is assumed to be a maximum estimator for $E\left(\bar{M}_{(n)}\right)$ and $n^{-1}\sum_{k=1}^{n} m_k$ to be a mean estimator for $E\left(\bar{M}_{(1)}\right)$. In this case, we have $\eta_1 = n$ and $\eta_2 = 1$. So, we can find from (30)

$$\hat{m}_{\max} = \frac{n+1}{n-1} m_n - \frac{2}{n-1}\frac{1}{n}\sum_{k=1}^{n} m_k = m_n + \frac{2}{n} m_n - \frac{2}{n-1}\frac{1}{n}\sum_{k=1}^{n-1} m_k,$$
$$\hat{m}_{\min} = -\frac{n+1}{n-1} m_n + n\frac{2}{n-1}\frac{1}{n}\sum_{k=1}^{n} m_k = -m_n + \frac{2}{n-1}\sum_{k=1}^{n-1} m_k.$$

The maximum or minimum could reach when $m_1 = m_2 = \cdots = m_{n-1}$. Thus, $\sum_{k=1}^{n-1} m_k = (n-1) m_1$ and

$$\begin{aligned} \hat{m}_{\max} &= m_n + \frac{2}{n}\left(m_n - m_1\right), \\ \hat{m}_{\min} &= m_1 - \left(m_n - m_1\right). \end{aligned} \tag{32}$$

We can see from here that

$$\begin{aligned} m_{\min} \le \hat{m}_{\max} &\le m_{\max} + \frac{2}{n}\left(m_{\max} - m_{\min}\right), \\ m_{\min} - \left(m_{\max} - m_{\min}\right) \le \hat{m}_{\min} &\le m_{\max}. \end{aligned} \tag{33}$$

If all events are equal, $m_1 = m_2 = \cdots = m_n$, we can see from (32) that $\hat{m}_{\max} = \hat{m}_{\min}$ indicating that the upper and lower limits are equal. In other words, if $m_{\min} = m_{\max}$ then $E\left(\bar{M}_{(\eta)}\right)$ is a constant (this case the distribution function is a delta function).

Asymmetrically distributed data

The case $\beta \neq 0$ is different, because the expected values are bounded also in the unbounded case of maximum as we will see later.

Let's assume that $\beta > 0$. We can use the KS-2 function

$$E\left(\bar{M}_{(\eta)}\right) = m_{\min} + \frac{\eta}{\beta}\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]\right)^k}{k\left(k+\eta\right)},$$

where $\eta \in \mathbb{R}_+$. When $\beta(m_{\max} - m_{\min}) = \infty$, we can get

$$m_{\min} = E\left(\bar{M}_{(\eta)}\right) - \frac{H_\eta}{\beta}, \tag{34}$$

where $H_\eta = \sum_{k=1}^{\infty} \eta / \left[k(k+\eta)\right]$ is a General Harmonic number (Abramowitz and Stegun, 1972; Haarala Orosco, 2016a).

The factor $\beta(m_{\max} - m_{\min})$ is unbounded if β, $m_{\max} - m_{\min}$ or both of them are unbounded. If we assume that β is bounded. The equation (34) shows that the $m_{\min}$ is bounded if and only if the expected value is bounded.

We can write the equation (34) of unbounded case, as

$$E\left(\bar{M}_{(\eta)}\right) = m_{\min} + \frac{H_\eta}{\beta}.$$

By reason of $f_\eta^{KS-2}\left(\beta(m_{\max} - m_{\min})\right) \geq 0$, when $0 \leq \beta(m_{\max} - m_{\min}) \leq \infty$, then $E\left(\bar{M}_{(\eta)}\right) \geq m_{\min}$. Thus, $m_{\min} \leq E\left(\bar{M}_{(\eta)}\right) \leq m_{\min} + H_\eta / \beta$ for any fixed $\eta \in \mathbb{R}_+$, no matter if the expected values $E\left(\bar{M}_{(\eta)}\right)$ are from bounded or unbounded data. If β is unbounded, then $E\left(\bar{M}_{(\eta)}\right) = m_{\min}$ for all $\eta \in \mathbb{N}$.

The same can also be shown when $\beta < 0$. Again, the factor $\beta(m_{\max} - m_{\min})$ is unbounded if β, $m_{\max} - m_{\min}$ or both of them are unbounded. In that case we will use the EKS-1 function because $\beta(m_{\max} - m_{\min}) \ll -\log(2)$. To find the limit in the unbounded case, we have

$$\begin{aligned}
E\left(\bar{M}_{(\eta)}\right) &= m_{\max} - \frac{1}{\beta} \frac{\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k+\eta} + \sum_{k=0}^{\infty} \frac{(-1)^k \left[\left(\exp\left[-\beta(m_{\max} - m_{\min})\right] - 1\right)^{-k+\eta} - 1\right]}{k-\eta}}{\left(\exp\left[-\beta(m_{\max} - m_{\min})\right] - 1\right)^{\eta}} \\
&= m_{\max} - \frac{1}{\beta} \Bigg\{ \left(\exp\left[-\beta(m_{\max} - m_{\min})\right] - 1\right)^{-\eta} \sum_{k=1}^{\infty} \frac{(-1)^k}{k+\eta} \\
&\qquad + \sum_{k=1}^{\infty} \frac{(-1)^k \left[\left(\exp\left[-\beta(m_{\max} - m_{\min})\right] - 1\right)^{-k} - \left(\exp\left[-\beta(m_{\max} - m_{\min})\right] - 1\right)^{-\eta}\right]}{k-\eta} \\
&\qquad + \frac{\left[1 - \left(\exp\left[-\beta(m_{\max} - m_{\min})\right] - 1\right)^{-\eta}\right]}{-\eta} \Bigg\}.
\end{aligned}$$

When $\beta(m_{\max} - m_{\min}) \to -\infty$, then $\left(\exp\left[-\beta(m_{\max} - m_{\min})\right] - 1\right)^{-\eta} \to 0$, $\left(\exp\left[-\beta(m_{\max} - m_{\min})\right] - 1\right)^{-k} \to 0$ and we get the expected value

$$E\left(\bar{M}_{(\eta)}\right) = m_{\max} + \frac{1}{\beta\eta} \tag{35}$$

and the maximum

$$m_{\max} = E\left(\bar{M}_{(\eta)}\right) - \frac{1}{\beta\eta}.$$

Similar way than before, $m_{\max}$ is bounded if and only if the expected value is bounded. Moreover, all the expected values from bounded or unbounded data are bounded because $m_{\max} + 1/(\beta\eta) \le E\left(\bar{M}_{(\eta)}\right) \le m_{\max}$ for fixed $\eta \in \mathbb{R}_+$. If β is unbounded, then $E\left(\bar{M}_{(\eta)}\right) = m_{\max}$ for all $\eta \in \mathbb{R}_+$.

The analysis above sounds quite theoretical. But we showed that if we have two distinct bounded estimators, $-\infty < \hat{m}_{(\eta_2)} \ne \hat{m}_{(\eta_1)} < \infty, \eta_2 < \eta_1$, then the β is bounded and at least one of the limits $m_{\min}$ or $m_{\max}$ is bounded. This means that at most the maximum $m_{\max}$ can be unbounded in the earthquake catalogue, where $\beta > 0$, and the minimum is bounded.

As we saw above, the expected values are bounded even the data is bounded or unbounded. This makes so difficult to estimate bounded $m_{\max}$. The recurrence formula gives one idea to show, if data is unbounded. It can find from the Appendix A.

The equation (34) can be written in the classical form

$$\beta = \frac{H_\eta}{E\left(\bar{M}_{(\eta)}\right) - m_{\min}}.$$

Owing to $E\left(\bar{M}_{(\eta)}\right) - m_{\min} > 0$ always, it implies the $\beta > 0$ for all $\eta \in \mathbb{R}_+$. This means, if data comes from unbounded system, that the *b*-value is always positive. It cannot get negative values.
Similar way, the equation (35) gives

$$\beta = -\frac{1}{\eta\left(m_{\max} - E\left(\bar{M}_{(\eta)}\right)\right)}.$$

Because of $m_{\max} - E\left(\bar{M}_{(\eta)}\right) > 0$, then $\beta < 0$ for all $\eta \in \mathbb{R}_+$.

We have shown above that the *b*-value do not change the sign if the data is unbounded. It means that getting positive and negative b-values within generalized estimators, is possible only in the case of bounded data and enough big η as we could see in figure 1.

Similar way as in the Uniform distribution case, we can create the new estimators. Let's consider the case $\beta > 0$. If $\eta_1, \eta_2 \in \mathbb{R}_+$, then

$$\beta = \frac{H_{\eta_1}}{E\left(\bar{M}_{(\eta_1)}\right) - m_{\min}}, \qquad \beta = \frac{H_{\eta_2}}{E\left(\bar{M}_{(\eta_2)}\right) - m_{\min}},$$

gives

$$m_{\min} = \frac{H_{\eta_2} E\left(\bar{M}_{(\eta_1)}\right) - H_{\eta_1} E\left(\bar{M}_{(\eta_2)}\right)}{H_{\eta_2} - H_{\eta_1}}, \qquad \beta = \frac{H_{\eta_2} - H_{\eta_1}}{E\left(\bar{M}_{(\eta_2)}\right) - E\left(\bar{M}_{(\eta_1)}\right)}.$$

This also shows that $m_{\min}$ and β are bounded and we can calculate them if there exist two different and bounded expected values.

In the case of $\beta<0$, $\eta_1,\eta_2\in\mathbb{R}_+$, we have

$$\beta=-\frac{1}{\eta_1\left(m_{\max}-E\left(\bar{M}_{(\eta_1)}\right)\right)},\qquad \beta=-\frac{1}{\eta_2\left(m_{\max}-E\left(\bar{M}_{(\eta_2)}\right)\right)}.$$

Thus,

$$m_{\max}=\frac{\eta_2\,E\left(\bar{M}_{(\eta_2)}\right)-\eta_1\,E\left(\bar{M}_{(\eta_1)}\right)}{\eta_2-\eta_1},\qquad \beta=-\frac{\eta_2-\eta_1}{\eta_1\eta_2\left(E\left(\bar{M}_{(\eta_2)}\right)-E\left(\bar{M}_{(\eta_1)}\right)\right)}.$$

In this case, $m_{\min}$ and β are bounded and possible to evaluate with two different and bounded expected values.

7. Expected value for the minimum

We will change the variable setting $\mathfrak{m}=-x+(m_{\max}+m_{\min})$, when $x_{\min}=m_{\max}$ and $x_{\max}=m_{\min}$. It implies that we flip the axes in such a way that the minimum will be the new maximum and the maximum will be the new minimum. Then the integral yields

$$\begin{aligned}
&\int_{m_{\min}}^{m_{\max}}\left(\frac{1-\exp\left[-(-\beta)(\mathfrak{m}-m_{\min})\right]}{1-\exp\left[-(-\beta)(m_{\max}-m_{\min})\right]}\right)^{\eta}d\mathfrak{m}\\
&\qquad=-\int_{x_{\min}}^{x_{\max}}\left(\frac{1-\exp\left[\beta(m_{\max}-x)\right]}{1-\exp\left[\beta(m_{\max}-m_{\min})\right]}\right)^{\eta}dx\\
&\qquad=-\int_{m_{\max}}^{m_{\min}}\left(\frac{\exp\left[-\beta(m_{\max}-m_{\min})\right]-1+1-\exp\left[-\beta(x-m_{\min})\right]}{\exp\left[-\beta(m_{\max}-m_{\min})\right]-1}\right)^{\eta}dx\\
&\qquad=\int_{m_{\min}}^{m_{\max}}\left(1-\frac{1-\exp\left[-\beta(x-m_{\min})\right]}{1-\exp\left[-\beta(m_{\max}-m_{\min})\right]}\right)^{\eta}dx\\
&\qquad=\int_{m_{\min}}^{m_{\max}}\left[1-F_M(m)\right]^{n}dm\,.
\end{aligned}\tag{36}$$

The KS-1 gives the minimum for $-\beta$. Having the expected value for the minimum, we have

$$\begin{aligned} E\left(\underline{M}_{(\eta)} \mid \beta\right) &= \int_{m_{\min}}^{m_{\max}} m\, d\left(1-\left[1-F_M(m)\right]^n\right) \\ &= m_{\max} - \int_{m_{\min}}^{m_{\max}} 1-\left[1-F_M(m)\right]^n dm \\ &= m_{\min} + \int_{m_{\min}}^{m_{\max}} \left[1-F_M(m)\right]^n dm \\ &= m_{\min} + \frac{1}{-\beta} f_\eta^{KS-1}\left(-\beta\left(m_{\max}-m_{\min}\right)\right). \end{aligned}$$

Taking into account that if $\beta(m_{\max}-m_{\min}) > \log(2)$, the KS-1 must be replaced by EKS-1. We see that the KS-1 function does not measure only the distance from the $m_{\max}$ to the expected value for the maximum, it is also a measure from the $m_{\min}$ to the expected value for the minimum with negative β.

We can see something more with these equations. Using (36), the expected value for the maximum can be considered as

$$\begin{aligned} E\left(\bar{M}_{(\eta)} \mid \beta\right) &= m_{\max} - \int_{m_{\min}}^{m_{\max}} \left[F_M(m \mid \beta)\right]^n dm \\ &= m_{\max} - \int_{m_{\min}}^{m_{\max}} \left[1-F_M(m \mid -\beta)\right]^n dm \\ &= E\left(\underline{M}_{(\eta)} \mid -\beta\right). \end{aligned}$$

In other words, the expected value curve for the maximum in the case of the positive β is equal than the expected value curve for the minimum in the case of negative β.

We saw that

$$\begin{aligned} E\left(\bar{M}_{(\eta)} \mid \beta\right) &= m_{\max} - \frac{1}{\beta} f_\eta^{KS-1}\left(\beta\left(m_{\max}-m_{\min}\right)\right), \\ E\left(\bar{M}_{(\eta)} \mid -\beta\right) &= m_{\min} + \frac{1}{-\beta} f_\eta^{KS-1}\left(-\beta\left(m_{\max}-m_{\min}\right)\right) = E\left(\underline{M}_{(\eta)} \mid \beta\right). \end{aligned}$$

These equations give straightforwardly

$$E\left(\underline{M}_{(\eta)} \mid \beta\right) = m_{\min} + \left(m_{\max} - E\left(\bar{M}_{(\eta)} \mid -\beta\right)\right). \quad (37)$$

This symmetry is easy to understand from Figure 3.

8. Examples

Figure 5 shows the behavior of the expected value curve with different b-values. All the curves start from the minimum the lower the value of b, the faster it reaches the maximum value.

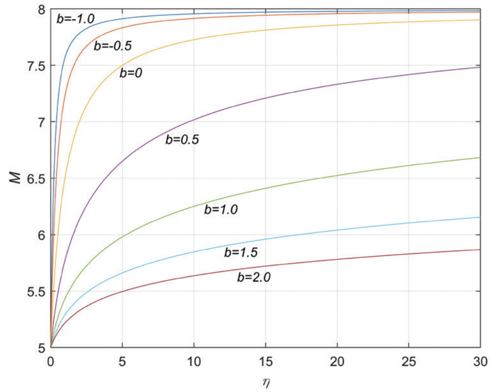

Figure 5. The expected value curves $E\left(\bar{M}_{(\eta)}\right)$

Figure 6 shows how the minimum and maximum follows the expected value curve in case of $b = 0.5$. The green and red lines present the expected values curves for minimum and maximum, respectively. The expected value curve for minimum is calculated by (37). The blue lines are acquired from the catalogues of the random samples. Each catalogue size η has generated a sample of 100 events, where the mean of maximums and mean of minimums are calculated from.

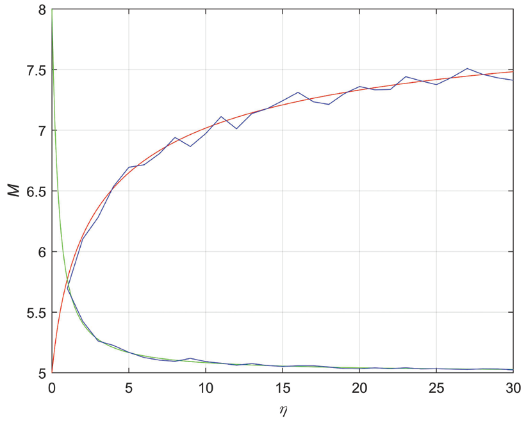

Figure 6. Mean of minimum and maximum for sample size 100 and b=0.5

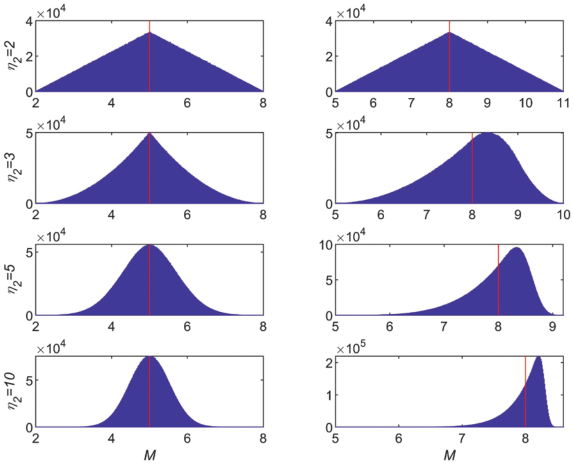

Figure 7. Distribution for minimum and maximum estimators in case of b=0

By comparison between Figures 5 and 6 we realize that the curves $E\left(\underline{M}_{(\eta)} \mid 0.5\right)$ and $E\left(\bar{M}_{(\eta)} \mid -0.5\right)$ are mirror images at the point $M = 6.5$.

Figure 7 shows some examples of the distribution for the maximum and minimum estimators $m_{\max}$ and $m_{\min}$ at (30), where we set

$$\hat{E}\left(\bar{M}_{(\eta_1)}\right) = \max_{1\le k\le \eta}\left(m_k\right),$$

$$\hat{E}\left(\bar{M}_{(\eta_2)}\right) = \frac{1}{n}\sum_{k=1}^{\eta} m_k \, .$$

The first is a maximum estimator and the second is a mean estimator. The left column of figures presents the distribution of the minimum estimators for cases of the catalogue size 2, 3, 5 and 10. Similar way, the right column of figures presents the distribution of the maximum estimators for cases of the catalogue size 2, 3, 5 and 10. The mount of catalogues in the sample are 10^7. The minimum, maximum, mean and the median for the sample of the minimum estimator are

$$\eta = 2: \begin{cases} \min\left(m_k\right) = 2.0005 \\ \max\left(m_k\right) = 7.9991 \\ \text{mean}\left(m_k\right) = 5.0005 \\ \text{median}\left(m_k\right) = 5.0008 \end{cases} \qquad \eta = 3: \begin{cases} \min\left(m_k\right) = 2.0203 \\ \max\left(m_k\right) = 7.9749 \\ \text{mean}\left(m_k\right) = 4.9997 \\ \text{median}\left(m_k\right) = 4.9997 \end{cases}$$

$$\eta = 5: \begin{cases} \min\left(m_k\right) = 2.1411 \\ \max\left(m_k\right) = 7.8534 \\ \text{mean}\left(m_k\right) = 5.0000 \\ \text{median}\left(m_k\right) = 4.9999 \end{cases} \qquad \eta = 10: \begin{cases} \min\left(m_k\right) = 2.4268 \\ \max\left(m_k\right) = 7.4739 \\ \text{mean}\left(m_k\right) = 5.0001 \\ \text{median}\left(m_k\right) = 5.0002 \end{cases}$$

The same statistic for the maximum estimators gives

$$\eta = 2: \begin{cases} \min(m_k) = 5.0013 \\ \max(m_k) = 10.9995 \\ \text{mean}(m_k) = 7.9999 \\ \text{median}(m_k) = 8.0000 \end{cases} \quad \eta = 3: \begin{cases} \min(m_k) = 5.0040 \\ \max(m_k) = 9.9890 \\ \text{mean}(m_k) = 7.9998 \\ \text{median}(m_k) = 8.1077 \end{cases}$$

$$\eta = 5: \begin{cases} \min(m_k) = 5.1414 \\ \max(m_k) = 9.1539 \\ \text{mean}(m_k) = 8.0000 \\ \text{median}(m_k) = 8.1121 \end{cases} \quad \eta = 10: \begin{cases} \min(m_k) = 5.7128 \\ \max(m_k) = 8.5374 \\ \text{mean}(m_k) = 7.9999 \\ \text{median}(m_k) = 8.0737 \end{cases}$$

We can see that the mean value gives the unbiased estimate for the maximum and minimum. In case on the minimum estimator, the median gives also unbiased estimate for the minimum because the distribution is symmetric. Moreover, the distributions are bounded with the limits (33).

9. Conclusion

We have given a general definition for the Gutenberg-Richter distribution function and the new series in the case of negative b-value. Moreover, we showed that if we have two bounded estimates for the expected values, then β is bounded and at least one of limits, $m_{\min}$ or $m_{\max}$, is bounded. We showed some results which gives the relation between positive and negative β. This work gives more perspective to understand the behavior of the Gutenberg-Richter distributed data.

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Appendix A

Reflection

In this section, we assume that $\eta \in \mathbb{R}_+ \setminus \mathbb{N}$. Let start with formula (18)

$$f_\eta^{EKS-1}\left(\beta\left(m_{\max}-m_{\min}\right)\right)=\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{-\eta}\sum_{k=1}^{\infty}\frac{(-1)^k}{k+\eta} + \sum_{k=0}^{\infty}\frac{(-1)^k\left[\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{-k}-\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{-\eta}\right]}{k-\eta}. \tag{38}$$

where $\infty < \beta\left(m_{\max}-m_{\min}\right) < -\log(2)$. It yields

$$\frac{\sum_{k=1}^{\infty}\frac{(-1)^k}{k+\eta}+\sum_{k=0}^{\infty}\frac{(-1)^k\left[\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{-k+\eta}-1\right]}{k-\eta}}{\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{\eta}}$$
$$=\frac{\sum_{k=1}^{\infty}\frac{(-1)^k}{k+\eta}-\sum_{k=1}^{\infty}\frac{(-1)^k}{k-\eta}+\frac{1}{\eta}+\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{\eta}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{-k}}{k-\eta}}{\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{\eta}}$$
$$=\frac{\frac{1}{\eta}-2\eta\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+\eta)(k-\eta)}}{\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{\eta}}-\frac{1}{\eta}+\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{-k}}{k-\eta}.$$

Using the cosine series (Abramowitz and Stegun, 1972)

$$\csc(z)=\frac{1}{z}+2z\sum_{k=1}^{\infty}\frac{(-1)^k}{z^2-k^2\pi^2}, \qquad z\neq k\pi,\ k\in\mathbb{Z}, \tag{39}$$

we get

$$\frac{1}{\eta}-2\eta\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2-\eta^2}=\frac{\pi}{\pi\eta}+\pi 2(\pi\eta)\sum_{k=1}^{\infty}\frac{(-1)^k}{(\pi\eta)^2-k^2\pi^2}=\pi\csc(\pi\eta). \tag{40}$$

If we set

$$1-\exp[-x]=\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{-1},$$

we find

$$x = -\log\left(1 - \frac{1}{1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]}\right).$$

Now, when $\infty < \beta\left(m_{\max} - m_{\min}\right) < -\log(2)$, then it is $-\log(2) < x < 0$ and we find

$$f_{-\eta}^{KS-1}(x) = f_{-\eta}^{KS-1}\left(-\log\left(1 - \frac{1}{1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]}\right)\right) = \sum_{k=1}^{\infty} \frac{\left(1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]\right)^{-k}}{k - \eta}. \qquad (41)$$

This shows that we could use the KS function also for the negative exponent. The results (40) and (41) let us to write (38) as

$$f_{\eta}^{EKS-1}\left(\beta\left(m_{\max} - m_{\min}\right)\right) = \frac{\pi \csc(\pi\eta)}{\left(\exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right] - 1\right)^{\eta}} - \frac{1}{\eta} + f_{-\eta}^{KS-1}\left(-\log\left(1 - \frac{1}{1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]}\right)\right).$$

Similarly, we can find the reflection formulae for the KS functions. For the KS-1 is

$$\begin{aligned} & f_{-\eta}^{KS-1}\left(\beta\left(m_{\max} - m_{\min}\right)\right) - f_{\eta}^{KS-1}\left(\beta\left(m_{\max} - m_{\min}\right)\right) \\ &= \sum_{k=1}^{\infty} \frac{\left(1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]\right)^{k}}{k - \eta} - \sum_{k=1}^{\infty} \frac{\left(1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]\right)^{k}}{k + \eta} \\ &= 2\eta \sum_{k=1}^{\infty} \frac{\left(1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]\right)^{k}}{(k - \eta)(k + \eta)} \qquad (42) \\ &= \frac{1}{\eta} - \frac{\pi}{\pi\eta} - \pi 2(\pi\eta) \sum_{k=1}^{\infty} \frac{\left(1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]\right)^{k}}{(\pi\eta)^2 - k^2\pi^2} \\ &= \frac{1}{\eta} - \pi \widetilde{\csc}_{\beta(m_{\max} - m_{\min})}(\pi\eta), \end{aligned}$$

where $\widetilde{\csc}_x(z)$ is called as a Generalized Cosine function (GC). The GC has the limits

$$\begin{aligned} \widetilde{\csc}_{-\log(2)}(z) &= \csc(z), \\ \widetilde{\csc}_{\infty}(z) &= \cot(z), \end{aligned}$$

because of (39) and (Abramowitz and Stegun, 1972)

$$\cot(z) = \frac{1}{z} + 2z \sum_{k=1}^{\infty} \frac{1}{z^2 - k^2\pi^2}, \qquad z \neq k\pi,\ k \in \mathbb{Z}.$$

Hence, we can write

$$\begin{aligned} f_{\eta}^{EKS-1}\left(\beta\left(m_{\max} - m_{\min}\right)\right) = f_{\eta}^{KS-1}&\left(-\log\left(1 - \frac{1}{1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]}\right)\right) \\ &+ \frac{\pi \csc(\pi\eta)}{\left(\exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right] - 1\right)^{\eta}} - \pi \widetilde{\csc}_{\beta(m_{\max} - m_{\min})}(\pi\eta). \end{aligned} \tag{43}$$

This is a reflection formula between EKS-1 and KS-1 functions. Moreover, because the EKS-1 and KS-1 are continuous functions, the subtraction

$$\frac{\pi \csc(\pi\eta)}{\left(\exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right] - 1\right)^{\eta}} - \pi \widetilde{\csc}_{\beta(m_{\max} - m_{\min})}(\pi\eta)$$

is bounded at the discontinuous points.

In a similar way as above, we can get the reflection formula for the KS-2 from (42) as

$$\begin{aligned} f_{-\eta}^{KS-1}\left(\beta\left(m_{\max} - m_{\min}\right)\right) - f_{\eta}^{KS-1}\left(\beta\left(m_{\max} - m_{\min}\right)\right) \\ = -f_{-\eta}^{KS-2}\left(\beta\left(m_{\max} - m_{\min}\right)\right) + f_{\eta}^{KS-2}\left(\beta\left(m_{\max} - m_{\min}\right)\right) \\ = \frac{1}{\eta} - \pi \widetilde{\csc}_{\beta(m_{\max} - m_{\min})}(\pi\eta). \end{aligned} \tag{44}$$

Because

$$f_{\eta}^{KS-2}(\infty) = \sum_{k=1}^{\infty} \frac{\eta}{k(k+\eta)} = \psi(1+\eta) + \gamma = H_{\eta},$$

where γ is a Euler constant and H_{η} is a General Harmonic number, the relation (44) gives a reflection formula of the Psi function (Abramowitz and Stegun, 1972)

$$\begin{aligned} \psi(1-\eta) &= \psi(1+\eta) - \frac{1}{\eta} + \pi \cot(\pi\eta) \\ &= \psi(\eta) + \pi \cot(\pi\eta). \end{aligned}$$

These reflection formulae (42)-(44) are not so nice in numerical calculus, even though they are possible to use. They become unstable in the neighborhood of the discontinuous point and we found that the formula (38) is quite powerful in the calculus having only one discontinuity point at $k=\eta$, which we do not need to consider in the case of $\eta > 20$.

Recurrence

The recurrence formula for the KS-1 function can be attained as

$$\begin{aligned} f_\eta^{KS-1}\left(\beta\left(m_{\max}-m_{\min}\right)\right) &= \sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^k}{k+\eta} \\ &= \sum_{k=0}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^k}{k+\eta}-\frac{1}{\eta} \\ &= \sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{k-1}}{k+\eta-1}-\frac{1}{\eta} \\ &= \frac{f_{\eta-1}^{KS-1}\left(\beta\left(m_{\max}-m_{\min}\right)\right)}{1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]}-\frac{1}{\eta}. \end{aligned}$$

or we can write it as

$$f_{\eta-1}^{KS-1}\left(\beta\left(m_{\max}-m_{\min}\right)\right)=\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)\left\{f_\eta^{KS-1}\left(\beta\left(m_{\max}-m_{\min}\right)\right)+\frac{1}{\eta}\right\}. \tag{45}$$

For the KS-2 function, the recurrence formula can be found as

$$\begin{aligned} f_\eta^{KS-2}\left(\beta\left(m_{\max}-m_{\min}\right)\right) &= \beta\left(m_{\max}-m_{\min}\right)-f_\eta^{KS-1}\left(\beta\left(m_{\max}-m_{\min}\right)\right) \\ &= \beta\left(m_{\max}-m_{\min}\right)-\frac{f_{\eta-1}^{KS-1}\left(\beta\left(m_{\max}-m_{\min}\right)\right)}{1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]}+\frac{1}{\eta} \\ &= \beta\left(m_{\max}-m_{\min}\right)-\frac{\beta\left(m_{\max}-m_{\min}\right)-f_{\eta-1}^{KS-2}\left(\beta\left(m_{\max}-m_{\min}\right)\right)}{1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]}+\frac{1}{\eta} \\ &= -\frac{\beta\left(m_{\max}-m_{\min}\right)\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]}+\frac{f_{\eta-1}^{KS-2}\left(\beta\left(m_{\max}-m_{\min}\right)\right)}{1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]}+\frac{1}{\eta}. \end{aligned}$$

This gives the recurrence formula for the Psi function (and General Harmonic number)

$$\begin{aligned} f_\eta^{KS-2}(\infty)=f_{\eta-1}^{KS-2}(\infty)+\frac{1}{\eta} \quad &\Leftrightarrow \quad H_\eta = H_{\eta-1}+\frac{1}{\eta} \\ &\Leftrightarrow \quad \psi(1+\eta)=\psi(\eta)+\frac{1}{\eta}. \end{aligned}$$

The recurrence formula (45) gives an interesting result for the maximum. Let's assume that $\beta > 0$. The formula (45) can be written now as

$$\beta\left(m_{\max} - E\left(\bar{M}_{(\eta-1)}\right)\right) = \left(1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]\right)\left\{\beta\left(m_{\max} - E\left(\bar{M}_{(\eta)}\right)\right) + \frac{1}{\eta}\right\}$$

or another way as

$$\frac{\beta\left(m_{\max} - E\left(\bar{M}_{(\eta-1)}\right)\right)}{\beta\left(m_{\max} - E\left(\bar{M}_{(\eta)}\right)\right) + \frac{1}{\eta}} = 1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]. \tag{46}$$

In the same way we have

$$\frac{\beta\left(m_{\max} - E\left(\bar{M}_{(\eta)}\right)\right)}{\beta\left(m_{\max} - E\left(\bar{M}_{(\eta+1)}\right)\right) + \frac{1}{\eta+1}} = 1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]. \tag{47}$$

We see from (46) and (47), that

$$\frac{\beta\left(m_{\max} - E\left(\bar{M}_{(\eta-1)}\right)\right)}{\beta\left(m_{\max} - E\left(\bar{M}_{(\eta)}\right)\right) + \frac{1}{\eta}} = \frac{\beta\left(m_{\max} - E\left(\bar{M}_{(\eta)}\right)\right)}{\beta\left(m_{\max} - E\left(\bar{M}_{(\eta+1)}\right)\right) + \frac{1}{\eta+1}}.$$

From this, we can find the maximum

$$m_{\max} = \frac{\eta E\left(\bar{M}_{(\eta-1)}\right)\left[\beta(\eta+1)E\left(\bar{M}_{(\eta+1)}\right) - 1\right] - (\eta+1)E\left(\bar{M}_{(\eta)}\right)\left[\beta\eta E\left(\bar{M}_{(\eta)}\right) - 1\right]}{1 + \beta\eta(\eta+1)\left(E\left(\bar{M}_{(\eta-1)}\right) - 2E\left(\bar{M}_{(\eta)}\right) + E\left(\bar{M}_{(\eta+1)}\right)\right)}$$

If all expected values $E\left(\bar{M}_{(\eta-1)}\right)$, $E\left(\bar{M}_{(\eta)}\right)$ and $E\left(\bar{M}_{(\eta+1)}\right)$ are bounded and the denominator is non-zero, the right-hand side is bounded showing that the $m_{\max}$ is bounded. This gives a condition

$$E\left(\bar{M}_{(\eta)}\right) \neq \frac{E\left(\bar{M}_{(\eta-1)}\right) + E\left(\bar{M}_{(\eta+1)}\right)}{2} + \frac{1}{2\beta\eta(\eta+1)}$$

for the bounded data. Thus, the upper bound of the data depends on the shape of the expected value curve $E\left(\bar{M}_{(\eta)}\right)$, $\eta \in \mathbb{R}_+$.

***Proof for the equation* (19)**

Let´s assume that $n \in \mathbb{N}$. Then (38) can be written as

$$f_n^{EKS-1}\left(\beta(m_{\max}-m_{\min})\right)=\left(\exp\left[-\beta(m_{\max}-m_{\min})\right]-1\right)^{-n}\times \left\{\sum_{k=1}^{\infty}\frac{(-1)^k}{k+n}+\sum_{\substack{k=0\\k\neq n}}^{\infty}\frac{(-1)^k\left[\left(\exp\left[-\beta(m_{\max}-m_{\min})\right]-1\right)^{-k+n}-1\right]}{k-n}-(-1)^n\log\left(\exp\left[-\beta(m_{\max}-m_{\min})\right]-1\right)\right\}. \quad (48)$$

We can see that

$$\sum_{\substack{k=0\\k\neq n}}^{\infty}\frac{(-1)^k\left[\left(\exp\left[-\beta(m_{\max}-m_{\min})\right]-1\right)^{-k+n}-1\right]}{k-n}$$
$$=\sum_{k=0}^{n-1}\frac{(-1)^k\left(\exp\left[-\beta(m_{\max}-m_{\min})\right]-1\right)^{-(k-n)}}{k-n}+\sum_{k=0}^{n-1}\frac{(-1)^k}{-k+n}$$
$$+(-1)^n\sum_{k=1}^{\infty}\frac{(-1)^k\left(\exp\left[-\beta(m_{\max}-m_{\min})\right]-1\right)^{-k}}{k}-(-1)^n\sum_{k=1}^{\infty}\frac{(-1)^k}{k}.$$

Firstly, we have

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k+n}+\sum_{k=0}^{n-1}\frac{(-1)^k}{-k+n}=\sum_{k=n+1}^{\infty}\frac{(-1)^{k-n}}{k}+\sum_{k=1}^{n}\frac{(-1)^{k-n}}{k}=(-1)^n\sum_{k=1}^{\infty}\frac{(-1)^k}{k}.$$

Secondly, we get

$$(-1)^n\sum_{k=1}^{\infty}\frac{(-1)^k\left(\exp\left[-\beta(m_{\max}-m_{\min})\right]-1\right)^{-k}}{k}=(-1)^n\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta(m_{\max}-m_{\min})\right]\right)^{-k}}{k}$$
$$=-(-1)^n\log\left(1-\frac{1}{1-\exp\left[-\beta(m_{\max}-m_{\min})\right]}\right)$$
$$=-(-1)^n\log\left(\frac{\exp\left[-\beta(m_{\max}-m_{\min})\right]}{\exp\left[-\beta(m_{\max}-m_{\min})\right]-1}\right)$$
$$=(-1)^n\beta(m_{\max}-m_{\min})+(-1)^n\log\left(\exp\left[-\beta(m_{\max}-m_{\min})\right]-1\right).$$

Thirdly, we can rewrite the partial sum as

$$\sum_{k=0}^{n-1}\frac{(-1)^k\left(\exp\left[-\beta(m_{\max}-m_{\min})\right]-1\right)^{-(k-n)}}{k-n}=-\sum_{k=1}^{n}\frac{(-1)^{n-k}\left(\exp\left[-\beta(m_{\max}-m_{\min})\right]-1\right)^k}{k}$$
$$=-(-1)^n\sum_{k=1}^{n}\frac{\left(1-\exp\left[-\beta(m_{\max}-m_{\min})\right]\right)^k}{k}.$$

Now, (48) yields

$$f_n^{EKS-1}\left(\beta(m_{\max}-m_{\min})\right)=\frac{(-1)^n\beta(m_{\max}-m_{\min})-(-1)^n\sum_{k=1}^{n}\frac{\left(1-\exp\left[-\beta(m_{\max}-m_{\min})\right]\right)^k}{k}}{\left(\exp\left[-\beta(m_{\max}-m_{\min})\right]-1\right)^n}$$
$$=\frac{\beta(m_{\max}-m_{\min})-\sum_{k=1}^{n}\frac{\left(1-\exp\left[-\beta(m_{\max}-m_{\min})\right]\right)^k}{k}}{\left(1-\exp\left[-\beta(m_{\max}-m_{\min})\right]\right)^n},$$

which was to proof.

Appendix B

We will show here an alternative way to solve the integral of the expected value for negative b. First of all, it is to change the variable as $\mathfrak{m}=-x$. It means that we flip the negative numbers to positive part and vice versa. The integral gives

$$\int_{m_{\min}}^{m_{\max}}\left(\frac{1-\exp\left[-\beta(\mathfrak{m}-m_{\min})\right]}{1-\exp\left[-\beta(m_{\max}-m_{\min})\right]}\right)^{\eta}d\mathfrak{m}=-\int_{-m_{\min}}^{-m_{\max}}\left(\frac{1-\exp\left[\beta\left(x-(-m_{\min})\right)\right]}{1-\exp\left[\beta\left((-m_{\max})-(-m_{\min})\right)\right]}\right)^{\eta}dx$$
$$=\int_{x_{\min}}^{x_{\max}}\left(\frac{1-\exp\left[-\beta(x_{\max}-x)\right]}{1-\exp\left[-\beta(x_{\max}-x_{\min})\right]}\right)^{\eta}dx$$
$$=\cdots$$

where $x_{\min}=-m_{\max}$ and $x_{\max}=-m_{\min}$. It means that $x_{\min}<x_{\max}$, because $m_{\min}<m_{\max}$. Multiplying the denominator and numerator by $\exp\left[\beta(x_{\max}-x_{\min})\right]$, we have

$$\cdots=\int_{x_{\min}}^{x_{\max}}\left(\frac{\exp\left[\beta(x_{\max}-x_{\min})\right]-\exp\left[\beta(x-x_{\min})\right]}{\exp\left[\beta(x_{\max}-x_{\min})\right]-1}\right)^{\eta}dx$$
$$=\int_{x_{\min}}^{x_{\max}}\left(\frac{\exp\left[\beta(x_{\max}-x_{\min})\right]-1+1-\exp\left[\beta(x-x_{\min})\right]}{\exp\left[\beta(x_{\max}-x_{\min})\right]-1}\right)^{\eta}dx$$
$$=\int_{x_{\min}}^{x_{\max}}\left(1-\frac{1-\exp\left[\beta(x-x_{\min})\right]}{1-\exp\left[\beta(x_{\max}-x_{\min})\right]}\right)^{\eta}dx.$$

Because $\beta < 0$, it is
$$-1 < -\frac{1-\exp\left[\beta\left(x-x_{\min}\right)\right]}{1-\exp\left[\beta\left(x_{\max}-x_{\min}\right)\right]} \le 0$$

when $\infty < \beta\left(x_{\max}-x_{\min}\right) < \beta\left(x-x_{\min}\right) \le 0$. Of course, this gives -1 at $x_{\max}$, but we do not consider it because the integral is the same, if we integrate over $\left[m_{\min}, m_{\max}\right]$ or $\left[m_{\min}, m_{\max}\right[$. We can apply now the Binomial Series (Abramowitz and Stegun, 1972) as

$$\begin{aligned}
\int_{x_{\min}}^{x_{\max}} \left(1-\frac{1-\exp\left[\beta\left(x-x_{\min}\right)\right]}{1-\exp\left[\beta\left(x_{\max}-x_{\min}\right)\right]}\right)^{\eta} dx &= \int_{x_{\min}}^{x_{\max}} \left\{\sum_{k=0}^{\infty}(-1)^k \binom{\eta}{k}\left(\frac{1-\exp\left[\beta\left(x-x_{\min}\right)\right]}{1-\exp\left[\beta\left(x_{\max}-x_{\min}\right)\right]}\right)^k\right\} dx \\
&= \sum_{k=0}^{\infty}(-1)^k \binom{\eta}{k}\left\{\int_{x_{\min}}^{x_{\max}}\left(\frac{1-\exp\left[-\left(-\beta\left(x-x_{\min}\right)\right)\right]}{1-\exp\left[-\left(-\beta\left(x_{\max}-x_{\min}\right)\right)\right]}\right)^k dx\right\} \qquad (49) \\
&= -\frac{1}{\beta}\sum_{k=0}^{\infty}(-1)^k \binom{\eta}{k} f_k^{KS-1}\left(-\beta\left(x_{\max}-x_{\min}\right)\right),
\end{aligned}$$

where

$$\binom{\eta}{0} = 1, \qquad \binom{\eta}{k} = \frac{\eta(\eta-1)(\eta-2)\cdots(\eta-k+1)}{k!}$$

and $f_0^{KS-1}\left(-\beta\left(x_{\max}-x_{\min}\right)\right) = -\beta\left(x_{\max}-x_{\min}\right)$. This is true, since

$$\int_{x_{\min}}^{x_{\max}}\left(\frac{1-\exp\left[\beta\left(x-x_{\min}\right)\right]}{1-\exp\left[\beta\left(x_{\max}-x_{\min}\right)\right]}\right)^0 dx = x_{\max}-x_{\min},$$

but also because of

$$\begin{aligned}
-\frac{1}{\beta} f_0^{KS-1}\left(-\beta\left(x_{\max}-x_{\min}\right)\right) &= -\frac{1}{\beta}\lim_{\eta\to 0+}\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[\beta\left(x_{\max}-x_{\min}\right)\right]\right)^k}{k+\eta} \\
&= -\frac{1}{\beta}\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[\beta\left(x_{\max}-x_{\min}\right)\right]\right)^k}{k} \\
&= \frac{1}{\beta}\log\left(1-\left(1-\exp\left[\beta\left(x_{\max}-x_{\min}\right)\right]\right)\right) \\
&= x_{\max}-x_{\min}.
\end{aligned}$$

The terms of the alternating series (49) are all positive term series of KS-1 functions, because of we have $\beta(x_{\max}-x_{\min})\le 0$. If we consider that $x_{\max}-x_{\min}=m_{\max}-m_{\min}$, we can write the final result as

$$\int_{m_{\min}}^{m_{\max}}\left(\frac{1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]}\right)^{\eta}d\mathfrak{m}=-\frac{1}{\beta}\sum_{k=0}^{\infty}(-1)^{k}\binom{\eta}{k}f_{k}^{KS-1}\left(-\beta\left(m_{\max}-m_{\min}\right)\right). \qquad (50)$$

It is interesting to see that we could carry the calculus from negative side to positive side. Anyway, this series is not so desirable because the magnitude of the binomial factor (49) increases quickly, and then decreases quickly, especially when η is big. Anyway, this relation (50) can be used only for small η because of the behavior of the binomial factor, but also because of the time to solve each KS function.

This kind of behavior of the factors is a problem. The binomial factor produces an overflow in the double precision system when η becomes big, for example,

$$\binom{1030}{515}>1.798\text{E}+308 \quad (= \text{maximum value in double precision})$$

meanwhile,

$$\binom{1030}{0}=\binom{1030}{1030}=1.$$

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